Geometric Spaces and Operations

Mathematical underpinnings of computer graphics

- Hierarchy of geometric spaces
  - Vector spaces
  - Affine spaces
  - Euclidean spaces
  - Cartesian spaces
  - Projective spaces
- Affine geometry and transformations
- Projective transformations and perspective
- Matrix formulations of transformations

Formally, a space is defined by

- A set of objects

- Operations on the objects
- Axioms defining invariant properties

Vector Spaces

Definition:

- Set of *vectors* $\mathcal{V}$
- Operations on $\vec{u},\ \vec{v} \in \mathcal{V}$:
  - *Addition:* $\vec{u} + \vec{v} \in \mathcal{V}$
  - *Scalar Multiplication:* $\alpha\vec{u} \in \mathcal{V}$ where $\alpha \in$ some field $\mathcal{F}$
- *Axioms*
  - *Unique zero element:* $0 + \vec{u} = \vec{u}$
  - *Field unit element:* $1\vec{u} = \vec{u}$
  - *Addition commutative:* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - *Addition associative:* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  - *Distributive scalar multiplication:* $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
- Additional definitions
  - Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.
  - Then $\mathcal{B}$ *spans* $\mathcal{V}$ iff any $\vec{v} \in \mathcal{V}$ can be written as $\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$.
  - $\sum_{i=1}^{n} \alpha_i \vec{v}_i$ is called a *linear combination* of the vectors in $\mathcal{B}$.
  - $\mathcal{B}$ is called a *basis* of $\mathcal{V}$ if it is a minimal spanning set.
  - All bases of $\mathcal{V}$ contain the same number of vectors.

  - The number of vectors in any basis of $\mathcal{V}$ is called the *dimension* of $\mathcal{V}$.
- Comments:
  - We are interested in 2 and 3 dimensional spaces.
  - No definition of distance (size) exists yet.
  - Angles and points have not been defined.

Affine Spaces

Definition:

- A set of vectors $\mathcal{V}$ and a set of *points* $\mathcal{P}$
- $\mathcal{V}$ is a vector space.
- Point-vector sum: $P + \vec{v} = Q$ with $P,\ Q \in \mathcal{P}$ and $\vec{v} \in \mathcal{V}$
- Additional definitions:
  - A *frame* $F = (\mathcal{B}, \mathcal{O})$ where $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis of $\mathcal{V}$ and the point $\mathcal{O}$ is called the *origin* of the frame.
  - The dimension of F is the same as the dimension of $\mathcal{V}$.
- Comments:
  - Still no distances or angles
  - Closer to what we want for graphics
  - The space has no distinguished origin

Euclidean Spaces

Definition:

- A *metric space* is any space with a *distance metric* $d(P, Q)$ defined on its elements.
- Distance metric axioms:
  - $d(P, Q) \geq 0$
  - $d(P, Q) = 0$ iff $P = Q$
  - $d(P, Q) = d(Q, P)$
  - $d(P, Q) \leq d(P, R) + d(R, Q)$ (triangle inequality)
- *Euclidean* distance metric:

$$d^2(P, Q) = (P - Q) \cdot (P - Q)$$

- Comments:
  - Euclidean metric based on dot product
  - Dot product defined on vectors
  - Distance metric defined on points
  - Distance is a property of the space, not a frame

- Dot product axioms:
  - $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
  - $\alpha(\vec{u} \cdot \vec{v}) = (\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Additional definitions:
  - The *norm* of a vector $\vec{u}$ is given by $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$.
  - Angles are defined by their cosines: $\cos(\angle\vec{u}\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$
  - Orthogonal vectors: $\vec{u} \cdot \vec{v} = 0 \rightarrow \vec{u} \perp \vec{v}$

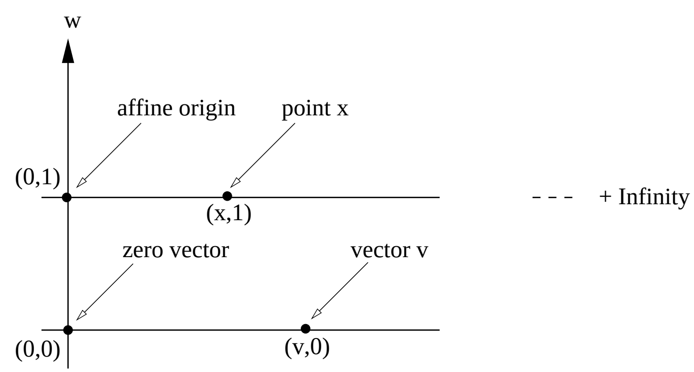
Cartesian Spaces

Definition:

- A frame $(\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$ is *orthonormal* iff
  - $\vec{i}, \vec{j}$, and $\vec{k}$ are *orthogonal*, i.e. $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ and
  - $\vec{i}, \vec{j}$, and $\vec{k}$ are *normal*, i.e. $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$
- Additional definitions:
  - The *standard frame* $F_s = (\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$
  - Points can be distinguished from vectors using an extra coordinate
    - 0 for vectors: $\vec{v} = (v_x, v_y, v_z, 0)$ means $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$
    - 1 for points: $P = (p_x, p_y, p_z, 1)$ means $P = p_x\vec{i} + p_y\vec{j} + p_z\vec{k} + \mathcal{O}$
- Comments
  - Coordinates have no meaning without an associated frame
  - There will be other ways to look at the extra coordinate
  - Sometimes we are sloppy and omit the extra coordinate
  - Assume standard frame unless specified otherwise
  - Points and vectors are different
  - Points and vectors have different operations
  - Points and vectors transform differently

Projective Space

- Affine Space = Vector Space + Points
- Projective Space = Affine Space + Infinity
- Homogeneous coordinate (x, y, z, w):
  - vector: $(x, y, z, 0)$
  - point: $(x, y, z, 1)$
- Embedding of vectors and points in space of one higher dimension



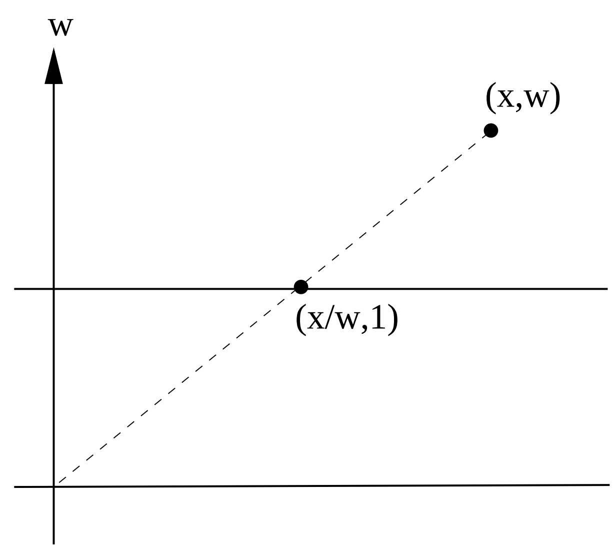
Projective Space

Projective Space:

- Division by the homogeneous coordinate
- Equivalence of affine points and homogeneous points
- Relationship with perspective
- More generally: rational splines

Projective Spaces (Augmented Affine Space)

- Divide through by homogenizing coordinate w
  $(x, w) \longrightarrow (\frac{x}{w}, 1)$
- All homogeneous points of the form $\alpha(x, 1), \alpha > 0$ are equivalent
- *Projects* homogeneous points *centrally* onto the affine plane



Comparisons

Affine Space	*Projective Space*
2D: (x, y)	(x, y, w)
3D: (x, y, z)	(x, y, z, w)
Two lines intersect if they are not parallel	Two lines always intersect

$$\begin{cases} 2x + 3y - 4 & = 0 \\ 4x + 6y - 9 & = 0 \end{cases}$$

never intersect	intersect at point at infinity $(2, 3, 0)$

A linear transformation can map:

an equilateral triangle to an isoceles triangle	a circle to a parabola

Homogeneous Coordinates

- Homogeneous coordinates represent n-space as a subspace of $n+1$ space
- For instance, homogeneous 4-space embeds ordinary 3-space as the $w = 1$ hyperplane
- Thus, we can obtain the 3-d image of any homogeneous point (wx, wy, wz, w), $w \neq 0$ as $(x, y, z, 1) = (wx/w, wy/w, wz/w, w/w)$, that is, by dividing all coordinates by w.
- Lines in homogeneous space which intersect the $w = 1$ hyperplane project to 3-space points.
- Notice that this is just a perspective projection from 4-d homogeneous space to 3-space, instead of dividing by z, we are dividing by w.

Relationship to Perspective:

- In rendering, the w values we generate are proportional to z
  - Equivalence corresponds to *perspective projection*

More Generally: Rational splines

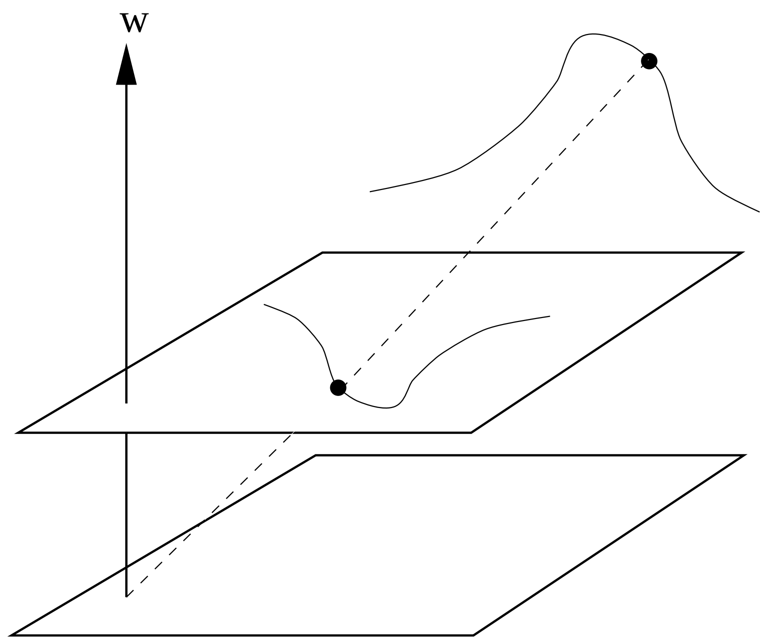
- Homogeneous spline curve
  - Spline curve of the form

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix} B_i^d(t) = \begin{bmatrix} \sum_{i=0}^{n} x_i B_i^d(t) \\ \sum_{i=0}^{n} y_i B_i^d(t) \\ \sum_{i=0}^{n} z_i B_i^d(t) \\ \sum_{i=0}^{n} w_i B_i^d(t) \end{bmatrix}$$

- Rational spline curve
  - Affine projective spline curve

$$\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} x_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \\ \sum_{i=0}^{n} y_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \\ \sum_{i=0}^{n} z_i B_i^d(t) / \sum_{i=0}^{n} w_i B_i^d(t) \end{bmatrix}$$

$$= \frac{1}{\sum_{i=0}^{n} w_i B_i^d(t)} \sum_{i=0}^{n} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} B_i^d(t)$$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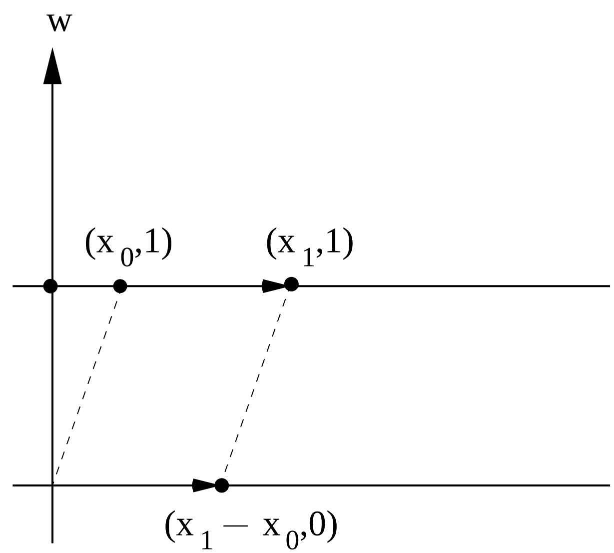
Vector and Affine Algebra

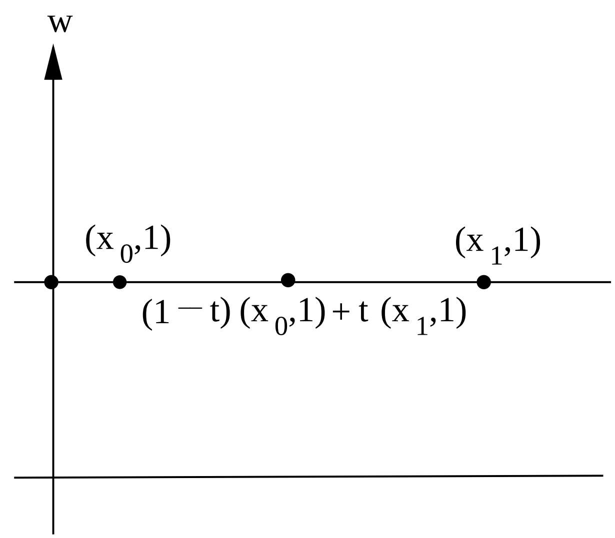
- Difference of points

$$(x_1, 1) - (x_0, 1) = (x_1 - x_0, 0)$$



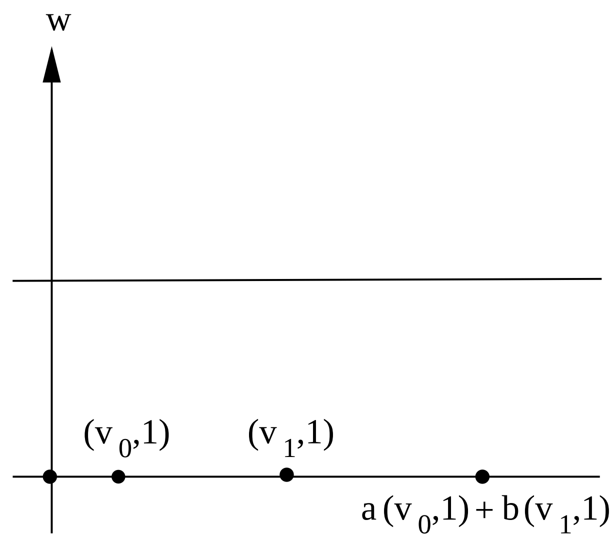
- Affine combination of points

$$(1-t)(x_1,1) + t(x_0,1) = ((1-t)x_1 + tx_0, 1)$$



- Linear combinations of vectors

$$a(v_0, 0) + b(v_1, 0) = (av_0 + bv_1, 0)$$



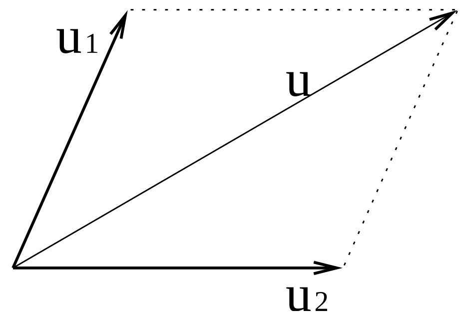
Linear Transformations

Vector space $\mathcal{V}$

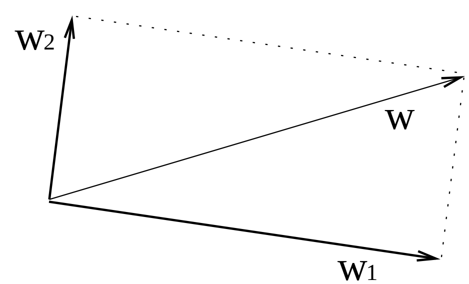
- Linear combinations of vectors in $\mathcal{V}$ are in $\mathcal{V}$
- For $\vec{u}, \vec{v} \in \mathcal{V}$
  - $\vec{u} + \vec{v} \in \mathcal{V}$
  - $\alpha\vec{u} \in \mathcal{V}$ for any scalar α
  - In general, $\sum_i \alpha_i \vec{u}_i \in \mathcal{V}$ for any scalars α_i
- Linear transformations
  - Let $\mathbf{T} : \mathcal{V}_0 \mapsto \mathcal{V}_1$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are vector spaces
  - Then $\mathbf{T}$ is *linear* iff
    - $\mathbf{T}(\vec{u} + \vec{v}) = \mathbf{T}(\vec{u}) + \mathbf{T}(\vec{v})$
    - $\mathbf{T}(\alpha\vec{u}) = \alpha\mathbf{T}(\vec{u})$
    - In general, $\mathbf{T}\left(\sum_i \alpha_i \vec{u}_i\right) = \sum_i \alpha_i \mathbf{T}\left(\vec{u}_i\right)$

Example of linear tranformation for vectors

$$u = \alpha_1 u_1 + \alpha_2 u_2$$



$$\begin{aligned} T(u) &= w \\ &= T(\alpha_1 u_1 + \alpha_2 u_2) \\ &= T(\alpha_1 u_1) + T(\alpha_2 u_2) \\ &= \alpha_1 T(u_1) + \alpha_2 T(u_2) \end{aligned}$$



Affine Transformations

Affine space $\mathcal{A} = (\mathcal{V}, \mathcal{P})$

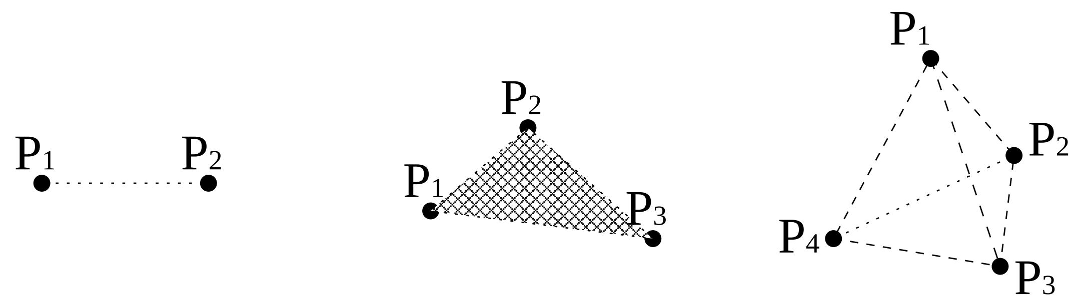
- For $\vec{u} \in \mathcal{V}$ and $P \in \mathcal{P}$

$$P + \vec{u} \in \mathcal{P}$$

- Define *point subtraction*:
  - For $P, Q \in \mathcal{P}$ and $\vec{u} \in \mathcal{V}$, if $P + \vec{u} = Q$, then $Q - P \equiv \vec{u}$
  - So in general we have $\sum_i \alpha_i P_i$ is a *vector* iff $\sum_i \alpha_i = 0$
- Define *point blending*:
  - For $P, P_1, P_2 \in \mathcal{P}$ and scalar α, if $P = P_1 + \alpha (P_2 - P_1)$ then $P \equiv (1 - \alpha) P_1 + \alpha P_2$
  - This can also be written $P \equiv \alpha_1 P_1 + \alpha_2 P_2$ where $\alpha_1 + \alpha_2 = 1$
  - So in general we have $\sum_i \alpha_i P_i$ is a *point* iff $\sum_i \alpha_i = 1$
- Geometrically, we have $\frac{|P-P_0|}{|P-P_1|} = \frac{d_1}{d_2}$ or $P = \frac{d_1 P_1 + d_2 P_2}{d_1 + d_2}$
- Vectors can always be combined linearly $\sum_i \alpha_i \vec{u}_i$
- Points can be combined linearly $\sum_i \alpha_i P_i$ iff
  - The coefficients sum to 1, giving a point ("affine combination")
  - The coefficients sum to 0, giving a vector ("vector combination")

– Example affine combination:

$$P(t) = P_0 + t(P_1 - P_0) = (1 - t)P_0 + tP_1$$



– This says any point on the line is an affine combination of the line segment's endpoints.

- Affine transformations
  - Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are affine spaces
  - $\mathbf{T}$ is said to be an *affine transformation* iff
    - $\mathbf{T}$ maps vectors to vectors and points to points
    - $\mathbf{T}$ is a linear transformation on the vectors
    - $\mathbf{T}(P + \vec{u}) = \mathbf{T}(P) + \mathbf{T}(\vec{u})$
  - Properties of affine transformations
    - $\mathbf{T}$ preserves affine combinations:

$$\mathbf{T}(\alpha_0 P_0 + \cdots + \alpha_n P_n) = \alpha_0 \mathbf{T}(P_0) + \cdots + \alpha_n \mathbf{T}(P_n)$$

where $\sum_i \alpha_i = 0$ or $\sum_i \alpha_i = 1$

* $\mathbf{T}$ maps lines to lines:

$$\mathbf{T}((1-t)P_0 + tP_1) = (1-t)\mathbf{T}(P_0) + t\mathbf{T}(P_1)$$

* $\mathbf{T}$ is affine iff it preserves ratios of distance along a line:

$$P = \frac{d_0 P_0 + d_1 P_1}{d_0 + d_1} \Rightarrow \mathbf{T}(P) = \frac{d_0 \mathbf{T}(P_0) + d_1 \mathbf{T}(P_1)}{d_0 + d_1}$$

* $\mathbf{T}$ maps parallel lines to parallel lines (can you prove this?)

– Example affine transformations
  * Rigid body motions (translations, rotations)
  * Scales, reflections
  * Shears

Reading Assignment and News

Chapter 4 pages 143 - 168, of Recommended Text.

Please also track the News section of the Course Web Pages for the most recent Announcements related to this course.

(http://www.cs.utexas.edu/users/bajaj/graphics23/cs354/)