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; This is the list of verification conditions for a FORTRAN square root program. For the details of the algorithm, see the comment at the end of the file. Boyer and Moore.

; This list of events has been further edited, for processing by DO-FILE, by (1) inserting the following NOTE-LIB, (2) commenting out each FORTRAN-COMMENT and following the comment with the corresponding macroexpansion, and (3) by commenting out each (COMMENT ...).

EVENT: Start with the library "fortran" using the compiled version.

; (FORTRAN-COMMENT FORTRAN)

Axiom: fortran

t
EVENT: Introduce the function symbol $i\theta$ of 0 arguments.

EVENT: Introduce the function symbol $isqrt1$ of 0 arguments.

EVENT: Introduce the function symbol $isqrt2$ of 0 arguments.

DEFINITION: $sq(i) = (i \ast i)$

AXIOM: input-conditions
\[ \texttt{\textquoteleft\texttt{*1\texttt{*true}}} \]

DEFINITION:

GLOBAL-HYPS
\[
= ((i\theta \in \texttt{N}) \\
\quad \wedge ((i\theta < \texttt{LEAST-INEXPRESSIBLE-POSITIVE-INTEGER}) \\
\quad \wedge \texttt{znumberp}(i\theta)))
\]

THEOREM: plus-1
\[(1 + x) = (1 + x)\]

THEOREM: difference-2
\[((1 + (1 + x)) - 2) = \texttt{fix}(x)\]

THEOREM: quotient-by-2
\[((a \div 2) + (a \div 2)) \not< (a - 1)\]

THEOREM: main-trick
\[sq(1 + ((j + k) \div 2)) \not< ((j \ast k) + j)\]

THEOREM: lessp-remainder2
\[((x \texttt{mod} y) < y) = (y \not\equiv 0)\]

THEOREM: remainder-quotient-elim
\[((y \not\equiv 0) \wedge (x \in \texttt{N})) \rightarrow (((x \texttt{mod} y) + (y \ast (x \div y))) = x)\]

THEOREM: sq-add1-non-zero
\[sq(1 + x) \not= 0\]

EVENT: Disable sq.

THEOREM: main
\[(j \not\equiv 0) \rightarrow (i < sq(1 + ((j + (i \div j)) \div 2)))\]

EVENT: Enable sq.
Theorem: lessp-times-cancellation-restated-for-linear
\((i \not< j) \rightarrow ((a \times i) \not< (a \times j))\)

Theorem: multiply-thru-by-divisor
\((a < (b \times c)) \rightarrow (((a \div b) < c) = t)\)

Theorem: times-greaterp-zero
\(((x \not= 0) \land (y \not= 0)) \rightarrow (0 < (x \times y))\)

Theorem: quotient-shrinks
\(i \not< (i \div j)\)

Theorem: quotient-shrinks-fast
\(i \not< (2 \times (i \div 2))\)

Theorem: quotient-by-1
\((i \div 1) = \text{fix}(i)\)

; (FORTRAN-COMMENT INPUT)

Axiom: input
\(t\)

; (FORTRAN-COMMENT LOGICAL-IF-T)

Axiom: logical-if-t
\(t\)

Theorem: stop
\((\neg \text{zlessp}(i0, i0)) \lor (\neg \text{GLOBAL-HYPS})\)

# | (COMMENT INPUT T) | #

Event: Undo back through the event named ‘logical-if-t’.

; (FORTRAN-COMMENT LOGICAL-IF-F)

Axiom: logical-if-f
\(t\)

; (FORTRAN-COMMENT LOGICAL-IF-T)

Axiom: logical-if-t
\(t\)
THEOREM: input-cond-of-zquotient
(zgreaterp (i$0, '1) ∧ GLOBAL-HYPS)
→ ((¬ zeqp ('2, '0)) ∧ expressible-znumberp (zquotient (i$0, '2)))

#| (COMMENT INPUT F T) |#

AXIOM: assignment
isqrt$1 = zquotient (i$0, '2)

THEOREM: lp
(zgreaterp (i$0, '1) ∧ GLOBAL-HYPS)
→ (((0 < isqrt$1)
    ∧ (i$0 < ('2 + isqrt$1))
    ∧ ((isqrt$1 ∈ N) ∧ (i$0 < sq (1 + isqrt$1))))
    ∧ lex (cons (isqrt$1, 'nil), cons (i$0, 'nil)))

#| (COMMENT INPUT F T) |#

EVENT: Undo back through the event named ‘logical-if-t’.

; (FORTRAN-COMMENT LOGICAL-IF-F1)

AXIOM: logical-if-f1

AxIOM: assignment
isqrt$1 = i$0

THEOREM: output
((¬ zgreaterp (i$0, '1)) ∧ GLOBAL-HYPS)
→ (znumberp (isqrt$1)
    ∧ (zgreatereqp (isqrt$1, '0)
        ∧ ((i$0 < sq (isqrt$1))
            ∧ (i$0 < sq ('1 + isqrt$1)))))

#| (COMMENT INPUT F F) |#

EVENT: Undo back through the event named ‘input’.

AXIOM: paths-from-lp

'1*true
**Definition:**

\[
\text{path-hyps} = (\text{global-hyps} \land ((\text{'}0 < \text{sqrt}$1)) \\
\quad \land ((\text{i}$0 \neq (\text{'}2 * \text{sqrt}$1)) \\
\quad \land ((\text{sqrt}$1 \in \mathbb{N}) \land (\text{i}$0 < \text{sq}(1 + \text{sqrt}$1)))))))
\]

**Theorem: definedness**

\[
\text{PATH-HYPS} \rightarrow \text{znumberp (sqrt$1)}
\]

#| (COMMENT LP) |#

**Theorem: input-cond-of-zquotient**

\[
\text{PATH-HYPS} \rightarrow ((\neg \text{zeqp (sqrt$1, '0)}) \\
\quad \land \text{expressible-znumberp (zquotient (i}$0, \text{sqrt}$1))}
\]

#| (COMMENT LP) |#

; (FORTRAN-COMMENT LOGICAL-IF-T)

**Axiom: logical-if-t**

\[
t
\]

**Theorem: output1**

\[
(\text{zgreatereqp (zquotient (i}$0, \text{sqrt}$1), \text{sqrt}$1) \land \text{PATH-HYPS}) \\
\rightarrow \text{znumberp (sqrt}$1) \\
\land \text{zgreatereqp (sqrt}$1, '0) \\
\land ((i}$0 \neq \text{sq (sqrt}$1)) \\
\land (i}$0 < \text{sq ('}1 + \text{sqrt}$1)))))
\]

#| (COMMENT LP T) |#

**Event: Undo back through the event named ‘logical-if-t’.

; (FORTRAN-COMMENT LOGICAL-IF-F2)

**Axiom: logical-if-f2**

\[
t
\]

**Theorem: input-cond-of-zquotient1**

\[
((\neg \text{zgreatereqp (zquotient (i}$0, \text{sqrt}$1), \text{sqrt}$1))) \land \text{PATH-HYPS}) \\
\rightarrow ((\neg \text{zeqp (sqrt}$1, '0)) \\
\land \text{expressible-znumberp (zquotient (i}$0, \text{sqrt}$1)))
\]
The correctness of the program depends upon the following events:
Definition.
(SQ I)
 =
(TIMES I I)

(FORTRAN-COMMENT ISQRT-STUFF)
Specification for routine ISQRT

The input assertion:
(AND (NUMBERP (I STATE))
 (LESSP (I STATE)
  (LEAST-INEXPRESSIBLE-POSITIVE-INTEGER)))

The output assertion:
(AND (ZGREATEREQP ANS 0)
 (NOT (LESSP (I STATE) (SQ ANS)))
 (LESSP (I STATE) (SQ (PLUS 1 ANS))))
INTEGER FUNCTION ISQRT(I)
INTEGER I
C CALCULATE THE SQUARE ROOT OF I USING THE NEWTON METHOD.
IF ((I .LT. 0)) STOP
IF ((I .GT. 1)) GOTO 100
ISQRT = I
RETURN
C ISQRT TAKES ON INCREASINGLY SMALLER VALUES AND CONVERGES TO THE SQ
C UARE ROOT OF I. THE FIRST APPROXIMATION IS ONE HALF I, WHICH IS NO
C T LESS THAN THE SQUARE ROOT OF I WHEN 1 IS LESS THAN I.
100 ISQRT = (I / 2)
200 CONTINUE
C ASSERTION LP
IF (((I / ISQRT) .GE. ISQRT)) RETURN
ISQRT = ((ISQRT + (I / ISQRT)) / 2)
C XXX SQ-REWRITE-OFF-AGAIN
GOTO 200
END

The XXX at ISQRT-STUFF.
@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. PLUS-1 (rewrite):
(EQUAL (PLUS 1 X) (ADD1 X))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. DIFFERENCE-2 (rewrite):
(EQUAL (DIFFERENCE (ADD1 (ADD1 X)) 2) (FIX X))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. QUOTIENT-BY-2 (rewrite):
(NOT (LESSP (PLUS (QUOTIENT A 2) (QUOTIENT A 2)) (SUB1 A)))
@END(VERBATIM)
Theorem. MAIN-TRICK (rewrite):

\[
\text{(NOT (LESSP (SQ (ADD1 (QUOTIENT (PLUS J K) 2)))
           (PLUS (TIMES J K) J)))}
\]

Hint: Induct as for (LESSP J K).

Theorem. LESSP-REMAINDER2 (rewrite and generalize):

\[
\text{(EQUAL (LESSP (REMAINDER X Y) Y)
           (NOT (ZEROP Y)))}
\]

Theorem. REMAINDER-QUOTIENT-ELIM (elimination):

\[
\text{(IMPLIES (AND (NOT (ZEROP Y)) (NUMBERP X))
             (EQUAL (PLUS (REMAINDER X Y)
                        (TIMES Y (QUOTIENT X Y)))
                     X))}
\]

Theorem. SQ-ADD1-NON-ZERO (rewrite):

\[
\text{(NOT (EQUAL (SQ (ADD1 X)) 0))}
\]

Enable SQ.
Theorem. MAIN (rewrite):
(IMPLIES (NOT (ZEROP J))
  (LESSP I
   (SQ (ADD1 (QUOTIENT (PLUS J (QUOTIENT I J))
         2))))))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. Disable SQ.
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. LESSP-TIMES-CANCELLATION-RESTATED-FOR-LINEAR (rewrite):
(IMPLIES (NOT (LESSP I J))
  (NOT (LESSP (TIMES A I) (TIMES A J))))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. MULTIPLY-THRU-BY-DIVISOR (rewrite):
(IMPLIES (LESSP A (TIMES B C))
  (EQUAL (LESSP (QUOTIENT A B) C) T))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. TIMES-GREATERP-ZERO (rewrite):
(IMPLIES (AND (NOT (ZEROP X)) (NOT (ZEROP Y)))
  (LESSP 0 (TIMES X Y)))
@END(VERBATIM)
@END(GROUP)

@BEGIN(GROUP)
@BEGIN(VERBATIM)
Theorem. QUOTIENT-SHRINKS (rewrite):
(NOT (LESSP I (QUOTIENT I J)))
@END(VERBATIM)
@END(GROUP)
Theorem. QUOTIENT-SHRINKS-FAST (rewrite):
\[ \text{(NOT (LESSP I (TIMES 2 (QUOTIENT I 2))))} \]

Theorem. QUOTIENT-BY-1 (rewrite):
\[ \text{(EQUAL (QUOTIENT I 1) (FIX I))} \]
Hints for routine ISQRT

The input clock:
   (LIST (I (START)))

The invariant and clock named LP.

   (AND (LESSP 0 (ISQRT STATE))
       (NOT (LESSP (I STATE)
                (TIMES 2 (ISQRT STATE))))
       (NUMBERP (ISQRT STATE))
       (LESSP (I STATE)
                (SQ (ADD1 (ISQRT STATE))))
   )
   (LIST (ISQRT STATE))

The XXX named SQ-REWRITE-OFF-AGAIN:
@BEGIN(GROUP)
@BEGIN(VERBATIM)
   Enable SQ.
@END(VERBATIM)
@END(GROUP)

|#
Index

assignment, 4
assignment1, 6

definedness, 5
difference-2, 2

expressible-znumberp, 4–6

fortran, 1

global-hyps, 2–5

i$0, 2–6
input, 3
input-cond-of-zplus, 6
input-cond-of-zquotient, 4, 5
input-cond-of-zquotient1, 5
input-cond-of-zquotient2, 6
input-conditions, 2
isqrt$1, 2, 4–6
isqrt$2, 2, 6

least-inexpressible-positive-integer, 2
lessp-remainder2, 2
lessp-times-cancellation-restate d-for-linear, 3
lex, 4, 6
logical-if-f, 3
logical-if-f1, 4
logical-if-f2, 5
logical-if-t, 3, 5
lp, 4
lp1, 6

main, 2
main-trick, 2
multiply-thru-by-divisor, 3

output, 4
output1, 5

path-hyps, 5, 6

paths-from-lp, 4
plus-1, 2

quotient-by-1, 3
quotient-by-2, 2
quotient-shrinks, 3
quotient-shrinks-fast, 3

remainder-quotient-elim, 2

sq, 2, 4–6
sq-add1-non-zero, 2
stop, 3

times-greaterp-zero, 3

zeqp, 4–6
zgreatereqp, 4–6
zgreaterp, 4
zlessp, 3
znumberp, 2, 4, 5
zplus, 6
zquotient, 4–6