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;;; Matt Kaufmann

EVENT: Start with the initial nfthm theory.

;;; Here's a little note showing a method for proving (in some cases)
;;; permutation-independence of list functions that are generated by
;;; associative-commutative binary functions. For example, we'd like to
;;; know that if SUMLIST is a function that adds up the elements of a
;;; given list, then permuting a list X doesn't change the value of
;;; SUMLIST(X). The method is as follows. First, we introduce a name
;;; AC-FN for an arbitrary associative-commutative binary function, along
;;; with the axioms saying so (and a "witness" for the consistency of this
;;; axiom, namely PLUS). This is followed by the definition of a function
;;; FOLDR, which is defined in terms of AC-FN by applying it repeatedly to
;;; the members of a given list. After defining PERMUTATION-P and proving
;;; some useful rewrite rules, I prove the main lemma FOLDR-PERMUTATION-P,
;;; which says (informally speaking, here) that FOLDR gives the same value
;;; when you permute its list argument. Then, using the "functional
;;; instantiation" mechanism in the Boyer-Moore system, I apply this
"generic" lemma (that is, FOLDR-PERMUTATION-P) to three examples: the
sum of the elements of a list, the product of the elements of a list,
and the WIRED-OR of the elements of a list (in a four-valued logic,
intuitively speaking, though I don’t actually ever need to say so).

As usual, this is all in Lisp syntax. Everything from a semicolon to
the end of a line is a comment, and I try to use lots of those in
order to explain what’s going on. Without further ado, then, here is
the annotated list of Boyer-Moore events (i.e. input).

By the way, it took about two hours for me to do this exercise
(including documentation). Replay time (real time) was about a minute
and a quarter on a Sun 3/60; run time reported was 24.8 secs. In case
it’s not clear.... the text below is all input to the Boyer-Moore
prover.

=================================================================

Add a new function declaring that the function ac-fn is
an associative-commutative binary function.

CONSERVATIVE AXIOM: ac-fn-intro
\[ ac-fn(x, y) = ac-fn(y, x) \]
\[ \land \ (ac-fn(ac-fn(x, y), z) = ac-fn(x, ac-fn(y, z))) \]

Simultaneously, we introduce the new function symbol ac-fn.

Next, recursively define a function that continually applies the
binary function AC-FN to the elements of a list. This is a
"fold-right" function; an analogous "fold-left" function exists,
and should be easy to prove equivalent to foldr; maybe I’ll do that
later.

DEFINITION:
foldr(lst, root)
= \text{if listp}(lst) \text{ then } ac-fn(car(lst), foldr(cdr(lst), root))
\text{else root endif}

The following function removes the first occurrence of the element
a from the list x; it’s auxiliary to the definition of permutation-p.

DEFINITION:
Here is the usual recursive definition of permutation-p.

**Definition:**

\[
\text{permutation-p}(x_1, x_2) =
\begin{cases}
\text{if} & \text{listp}(x_1) \\
\quad \text{then} & (\text{car}(x_1) \in x_2) \land \text{permutation-p}(\text{cdr}(x_1), \text{remove1}(\text{car}(x_1), x_2)) \\
\quad \text{else} & x_2 \equiv \text{nil}
\end{cases}
\]

The strategy below is as follows. I wanted to prove that foldr is preserved under permutations of its (first) argument; that’s the lemma called FOLDR-PERMUTATION-P below. The proof attempt led me to prove a lemma called FOLDR-REMOVE1, which occurs just above FOLDR-PERMUTATION-P. In order to prove FOLDR-REMOVE1, though, I found that I needed a property of associative-commutative functions, stated in the lemma AC-FN-COMMUTATIVITY-2 below.

The following two lemmas are used in the proof of the lemma named AC-FN-COMMUTATIVITY-2 below, which is key to FOLDR-REMOVE1, which in turn is crucial for FOLDR-PERMUTATION-P.

**Theorem:** ac-fn-assoc-reversed

\[
\text{ac-fn}(x, \text{ac-fn}(y, z)) = \text{ac-fn}(\text{ac-fn}(x, y), z)
\]

**Theorem:** ac-fn-comm

\[
\text{ac-fn}(x, z) = \text{ac-fn}(z, x)
\]

**Theorem:** ac-fn-commutativity-2

\[
\text{ac-fn}(z, \text{ac-fn}(x, a)) = \text{ac-fn}(x, \text{ac-fn}(z, a))
\]

The lemma AC-FN-ASSOC-REVERSED was used in the proof of the lemma immediately above, but now we want to turn it off as a rewrite rule so that it doesn’t loop in combination with the associativity rule introduced at the outset.

**Event:** Disable ac-fn-assoc-reversed.

**Theorem:** foldr-remove1

\[
(z \in x_2) \rightarrow (\text{ac-fn}(z, \text{foldr}(\text{remove1}(z, x_2), \text{root}))) = \text{foldr}(x_2, \text{root})
\]
Theorem: foldr-permutation-p
\[ \text{permutation-p}(x_1, x_2) \rightarrow (\text{foldr}(x_1, \text{root}) = \text{foldr}(x_2, \text{root})) \]

;; Having proved this general fact about foldr, let us apply it to
;; three examples: the sum of the elements of a list, the product
;; of the elements of a list, and a wired-or function.

;; SUMLIST ;;

;; First, we give a natural recursive definition of the sum of the
;; elements of a list. One could easily generate such definitions
;; automatically from the definition of foldr, by the way; for now,
;; I’ll take each application as it comes.

Definition:
\[
\text{sumlist}(\text{lst}) \quad = \quad \text{if } \text{listp}(\text{lst}) \text{ then } \text{car}(\text{lst}) + \text{sumlist}(\text{cdr}(\text{lst})) \quad \text{else } 0 \text{ endif}
\]

;; Let us now instantiate the main result called FOLDR-PERMUTATION-P
;; above to the particular case in question, i.e. to the case of the
;; sum of the elements of a list.

Theorem: sumlist-permutation-p-lemma
\[*auto*

;; Finally, I’ll use the lemma above as a hint so that the theorem that
;; SUMLIST is invariant under a permutation of its argument is immediate.

Theorem: sumlist-permutation-p
\[ \text{permutation-p}(x_1, x_2) \rightarrow (\text{sumlist}(x_1) = \text{sumlist}(x_2)) \]

;; TIMESLIST ;;

;; Now let’s repeat the exercise above for TIMES. This case proceeds
;; similarly to the PLUS case, except we need a few lemmas about TIMES
;; because less is built into the prover about TIMES than for PLUS.
;; In practice, many users of the prover at CLInc would load a
;; standard arithmetic library that has these facts about TIMES, any
;; many others, included in it. (Such a library will have already
;; been proved correct, so such an inclusion is sound.)
Theorem: times-assoc
\((x \ast y) \ast z = (x \ast (y \ast z))\)

Theorem: times-1
\((x \ast 1) = \text{fix}(x)\)

Theorem: times-comm
\((x \ast z) = (z \ast x)\)

;; Now we repeat the three main events that we did for PLUS:
;; definition of the n-ary version, the functional instantiation, and
;; the main result. It turns out that we need the "commutativity-2"
;; property proved above for ac-fn as a lemma; the first
;; functionally-instantiate event below derives this property for
;; times as an immediate corollary.

Definition:
timeslist \((lst)\)
\(= \text{if listp}(lst) \text{ then } \text{car}(lst) \ast \text{timeslist}(\text{cdr}(lst))\)
\(
\text{else } 1 \text{ endif}
\)

;; Need commutativity-2 as a lemma.....

Theorem: times-commutativity-2
*auto*

Theorem: timeslist-permutation-p-lemma
*auto*

Theorem: timeslist-permutation-p
permutation-p \((x_1, x_2) \rightarrow (\text{timeslist}(x_1) = \text{timeslist}(x_2))\)

;;;; WIRED-OR ;;;;;

;; Let’s say that wired-or treats Z as an identity, and returns X if
;; either argument is not Z. In particular, the OR of Z with itself
;; is Z.

;; First, the binary version.....

Definition:
\(\text{or2}(a, b)\)
\(= \text{if } a = 'z \text{ then } b\)
\(\text{elseif } b = 'z \text{ then } a\)
\(\text{else } 'x \text{ endif}\)
DEFINITION:
wired-or (lst)
= if listp (lst) then or2 (car (lst), wired-or (cdr (lst)))
else 'z endif

;; Now we just copy the usual two events, using 'z for root.
;; Commutativity-2 should be trivial in this case, so I won't separate
;; it out as a separate lemma as I did for the TIMES version above.

THEOREM: wired-or-permutation-p-lemma
*auto*

THEOREM: wired-or-permutation-p
permutation-p (x1, x2) → (wired-or (x1) = wired-or (x2))
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