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|#

;; Matt Kaufmann

;; Koenig Tree Lemma events supporting ``An Extension of the
;; Boyer-Moore Theorem Prover to Support First-Order Quantification,''
;; to appear in JAR (1992?).  The DEFN-SK events have been replaced by
;; DCLs and ADD-AXIOMs, as shown.

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;; Koenig's Lemma Events List
;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;; All this initial stuff is just to get the CONSTRAIN below accepted.
```

EVENT: Start with the initial **nqthm** theory.

DEFINITION:
ones (n)
$=$ **if** $n \simeq 0$ **then nil**
else cons $(1,$ ones $(n-1))$ **endif**

DEFINITION:
all-ones (s)
= **if** listp (s) **then** $(\text{car}\,(s) = \texttt{1}) \wedge$ all-ones $(\text{cdr}\,(s))$
else $s =$ **nil endif**

DEFINITION:
length (s)
= **if** listp (s) **then** $1 +$ length $(\text{cdr}\,(s))$
else `0` **endif**

DEFINITION:
subseq $(s1,\ s2)$
= **if** $s1 = s2$ **then t**
elseif $s2 \simeq$ **nil then f**
else subseq $(s1,\ \text{cdr}\,(s2))$ **endif**

THEOREM: subseq-all-ones
$(\text{all-ones}\,(s1) \wedge \text{subseq}\,(s2,\ s1)) \rightarrow \text{all-ones}\,(s2)$

DEFINITION:
plistp (s)
= **if** listp (s) **then** plistp $(\text{cdr}\,(s))$
else $s =$ **nil endif**

THEOREM: plistp-all-ones
$\text{all-ones}\,(s) \rightarrow \text{plistp}\,(s)$

THEOREM: all-ones-ones
$\text{all-ones}\,(\text{ones}\,(n))$

THEOREM: ones-is-injective
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (i \neq j)) \rightarrow (\text{ones}\,(i) \neq \text{ones}\,(j))$

CONSERVATIVE AXIOM: koenig-intro
node-p (**nil**)
$\wedge$ (truep (node-p (s)) $\vee$ falsep (node-p (s)))
$\wedge$ (node-p (s)
$\rightarrow$ (node-p (cons $(n,\ s)$) $= ((\texttt{0} < n) \wedge (\text{succard}\,(s) \not< n))$))
$\wedge$ ((node-p $(s1)$ $\wedge$ subseq $(s,\ s1)$) $\rightarrow$ node-p (s))
$\wedge$ node-p (s-n (n))
$\wedge$ $(((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (i \neq j)) \rightarrow (\text{s-n}\,(i) \neq \text{s-n}\,(j)))$
$\wedge$ $((\neg\ \text{plistp}\,(s)) \rightarrow (\neg\ \text{node-p}\,(s)))$

Simultaneously, we introduce the new function symbols *node-p*, *succard*, and *s-n*.

```
;; We want to define a function s-height which returns an element of a given height.
;; The next several events culminate in the following lemma:

;; (prove-lemma length-s-height (rewrite)
;;   (equal (length (s-height n)) (fix n)))
```

DEFINITION:
succ-aux $(s,\ n)$
= **if** $n \simeq$ `0` **then nil**
else cons (cons $(n,\ s)$, succ-aux $(s,\ n-1)$) **endif**

DEFINITION: successors (s) = succ-aux $(s,$ succard $(s))$

DEFINITION:
successors-list (ss)
= **if** listp (ss)
then append (successors (car (ss)), successors-list (cdr (ss)))
else nil endif

DEFINITION:
level (n)
= **if** $n \simeq$ `0` **then** list (**nil**)
else successors-list (level $(n-1)$) **endif**

DEFINITION:
init-tree (n)
= **if** $n \simeq$ `0` **then** list (**nil**)
else append (level (n), init-tree $(n-1)$) **endif**

DEFINITION:
remove1 $(a,\ x)$
= **if** listp (x)
then if $a =$ car (x) **then** cdr (x)
else cons (car (x), remove1 $(a,$ cdr $(x))$) **endif**
else x **endif**

THEOREM: length-remove1
$(a \in x) \rightarrow$ (length (remove1 $(a,\ x)$) $<$ length (x))

DEFINITION:
first-non-member-index $(i,\ x)$
= **if** s-n $(i) \in x$ **then** first-non-member-index $(1+i,$ remove1 (s-n (i), $x))$
else i **endif**

DEFINITION:
nthcdr $(n,\ s)$
$=$ **if** $n \simeq \mathtt{0}$ **then** s
else nthcdr $(n - 1,$ cdr $(s))$ **endif**

DEFINITION:
s-height (n)
$=$ nthcdr (length (s-n (first-non-member-index ($\mathtt{0}$, init-tree (n)))) $- n$,
s-n (first-non-member-index ($\mathtt{0}$, init-tree (n))))

THEOREM: nthcdr-subseq
(length $(s) \not< n) \rightarrow$ subseq (nthcdr $(n,\ s),\ s)$

THEOREM: node-p-nthcdr
(node-p $(s) \wedge$ (length $(s) \not< n)) \rightarrow$ node-p (nthcdr $(n,\ s)$)

THEOREM: lessp-difference-1
$(x < (x - n)) = \mathbf{f}$

THEOREM: node-p-s-height
node-p (s-height (n))

THEOREM: length-nthcdr
length (nthcdr $(n,\ s)) =$ (length $(s) - n)$

THEOREM: first-non-member-index-lessp
first-non-member-index $(i,\ x) \not< i$

THEOREM: s-n-first-non-member-index-not-equal
$(i \in \mathbf{N})$
$\rightarrow$ (s-n (first-non-member-index $(1 + i,$ remove1 (s-n $(i),\ x))) \neq$ s-n $(i))$

THEOREM: member-remove1
$(a \neq b) \rightarrow ((a \in$ remove1 $(b,\ x)) = (a \in x))$

THEOREM: s-n-first-non-member-index
$(i \in \mathbf{N}) \rightarrow$ (s-n (first-non-member-index $(i,\ x)) \notin x)$

THEOREM: member-append
$(a \in$ append $(x,\ y)) = ((a \in x) \vee (a \in y))$

THEOREM: member-cons-succ-aux
(cons $(z,\ v) \in$ succ-aux $(v,\ n)) = ((\mathtt{0} < z) \wedge (n \not< z))$

THEOREM: node-p-cons-lemma
$(\neg$ node-p $(s)) \rightarrow (\neg$ node-p (cons $(n,\ s)))$

THEOREM: node-p-cons
node-p (cons (*n*, *s*)) = (node-p (*s*) ∧ (0 < *n*) ∧ (succard (*s*) ≮ *n*))

DEFINITION:
all-length-n (*ss*, *n*)
= **if** listp (*ss*) **then** (length (car (*ss*)) = *n*)
∧ all-length-n (cdr (*ss*), *n*)
else t endif

THEOREM: all-length-n-append
all-length-n (append (*ss1*, *ss2*), *n*)
= (all-length-n (*ss1*, *n*) ∧ all-length-n (*ss2*, *n*))

THEOREM: all-length-n-succ-aux
(length (*s*) = *n*) → all-length-n (succ-aux (*s*, *k*), 1 + *n*)

THEOREM: all-length-n-successors-list
all-length-n (*ss*, *n*) → all-length-n (successors-list (*ss*), 1 + *n*)

THEOREM: length-0
(length (*s*) = 0) = (*s* ≃ **nil**)

DEFINITION:
member-level-induction (*s*, *n*)
= **if** *n* ≃ 0 **then t**
else member-level-induction (cdr (*s*), *n* − 1) **endif**

THEOREM: succ-aux-listp
(¬ listp (*s*)) → (*s* ∉ succ-aux (*z*, *n*))

THEOREM: successors-list-listp
(¬ listp (*s*)) → (*s* ∉ successors-list (*ss*))

THEOREM: member-succ-aux
(*s* ∈ succ-aux (*x*, *n*)) → (cdr (*s*) = *x*)

THEOREM: member-successors-list-successors-list-witness
(*s* ∈ successors-list (*ss*))
= ((cdr (*s*) ∈ *ss*) ∧ (*s* ∈ successors (cdr (*s*))))

THEOREM: member-level
((*n* ∈ **N**) ∧ node-p (*s*)) → ((*s* ∈ level (*n*)) = (length (*s*) = *n*))

THEOREM: member-init-tree
node-p (*s*) → ((*s* ∈ init-tree (*n*)) = (*n* ≮ length (*s*)))

THEOREM: length-s-non-member-index
$(i \in \mathbf{N}) \rightarrow (n < \text{length}\,(\text{s-n}\,(\text{first-non-member-index}\,(i, \text{init-tree}\,(n)))))$

THEOREM: length-s-height
$\text{length}\,(\text{s-height}\,(n)) = \text{fix}\,(n)$

EVENT: Disable s-height.

```
;; End of s-height excursion.

;; Our goal:

;(prove-lemma konig-tree-lemma nil
;  (and (node-p (k n))
;       (implies (not (lessp j i))
;                (subseq (k i) (k j)))
;       (equal (length (k n)) (fix n))))

#| The original DEFN-SK event here was processed as follows:

>(defn-sk inf (s)
  ;; says that s has arbitrarily high successors
  (forall big-h (exists big-s
    (and (subseq s big-s)
         (node-p big-s)
         (lessp big-h (length big-s))))))


Adding the Skolem axiom:
      (AND (IMPLIES (AND (SUBSEQ S BIG-S)
                         (NODE-P BIG-S)
                         (LESSP (BIG-H S) (LENGTH BIG-S)))
                    (INF S))
           (IMPLIES (NOT (AND (SUBSEQ S (BIG-S BIG-H S))
                              (NODE-P (BIG-S BIG-H S))
                              (LESSP BIG-H
                                     (LENGTH (BIG-S BIG-H S)))))
                    (NOT (INF S)))).


     As this is a DEFN-SK we can conclude that:
      (OR (TRUEP (INF S)) (FALSEP (INF S)))
is a theorem.
```

```
[ 0.2 0.0 0.0 ]
INF

>
|#
```

EVENT: Introduce the function symbol *inf* of one argument.

EVENT: Introduce the function symbol *big-h* of one argument.

EVENT: Introduce the function symbol *big-s* of 2 arguments.

AXIOM: inf-intro
$((\text{subseq}(s, \textit{big-s}) \wedge \text{node-p}(\textit{big-s}) \wedge (\text{big-h}(s) < \text{length}(\textit{big-s})))$
$\rightarrow \quad \text{inf}(s))$
$\wedge \quad ((\neg (\text{subseq}(s, \text{big-s}(\textit{big-h}, s))$
$\wedge \quad \text{node-p}(\text{big-s}(\textit{big-h}, s))$
$\wedge \quad (\textit{big-h} < \text{length}(\text{big-s}(\textit{big-h}, s)))))$
$\rightarrow \quad (\neg \text{inf}(s)))$

AXIOM: inf-boolean
$\text{truep}(\text{inf}(s)) \vee \text{falsep}(\text{inf}(s))$

```
;; The following three events were generated mechanically.  They are
;; useful especially for applying the Skolem axioms for INF inside the
;; proof-checker, via the macro command SK*.
```

EVENT: Disable inf-intro.

THEOREM: inf-suff
$(\text{subseq}(s, \textit{big-s}) \wedge \text{node-p}(\textit{big-s}) \wedge (\text{big-h}(s) < \text{length}(\textit{big-s})))$
$\rightarrow \quad \text{inf}(s)$

THEOREM: inf-necc
$(\neg (\text{subseq}(s, \text{big-s}(\textit{big-h}, s))$
$\wedge \quad \text{node-p}(\text{big-s}(\textit{big-h}, s))$
$\wedge \quad (\textit{big-h} < \text{length}(\text{big-s}(\textit{big-h}, s)))))$
$\rightarrow \quad (\neg \text{inf}(s))$

DEFINITION:
next $(s,\ max)$
$=$ **if** $max \simeq \mathtt{0}$ **then** cons $(\mathtt{0},\ s)$
elseif inf (cons $(max,\ s)$) **then** cons $(max,\ s)$
else next $(s,\ max - 1)$ **endif**

```
;; We want to show that NEXT gives us a successor with infinitely many
;; successors.

; INF-IMPLIES-INF-NEXT:
;(implies (and (node-p s)
;              (inf s))
;         (inf (next s (succard s))))

;; Note that if some successor of s has infinitely many successors, so
;; does (NEXT S (SUCCARD S)).  This is the lemma
;; INF-CONS-IMPLIES-INF-NEXT below.  But first note:
```

THEOREM: inf-implies-node-p
inf $(s) \rightarrow$ node-p (s)

THEOREM: not-inf-zerop
$(i \simeq \mathtt{0}) \rightarrow (\neg$ inf (cons $(i,\ s)))$

THEOREM: inf-cons-implies-inf-next
(node-p $(s) \wedge$ inf (cons $(i,\ s)) \wedge (n \not< i)) \rightarrow$ inf (next $(s,\ n)$)

```
;; Our goal now is to apply this lemma by proving that
;; (inf (cons i s)) for some i <= (succard s).
```

DEFINITION:
all-big-h $(s,\ n)$
$=$ **if** $n \simeq \mathtt{0}$ **then** $1 +$ length (s)
else big-h (cons $(n,\ s)) +$ all-big-h $(s,\ n - 1)$ **endif**

THEOREM: all-big-h-length
length $(s) <$ all-big-h $(s,\ n)$

THEOREM: all-big-h-lessp
$((\mathtt{0} < i) \wedge (n \not< i))$
$\rightarrow$ ((big-h (cons $(i,\ s)) <$ all-big-h $(s,\ n)) = \mathbf{t}$)

```
;; Here's a function which tells us which way s first branches on its
;; way to extending to s1.
```

DEFINITION:
first-branch $(s,\ s1)$
$=$ **if** $s = \mathrm{cdr}\,(s1)$ **then** $\mathrm{car}\,(s1)$
elseif $s1 \simeq$ **nil then** 0
else first-branch $(s,\ \mathrm{cdr}\,(s1))$ **endif**

THEOREM: subseq-cons-first-branch
$(\mathrm{subseq}\,(s,\ x) \wedge (s \neq x)) \rightarrow \mathrm{subseq}\,(\mathrm{cons}\,(\text{first-branch}\,(s,\ x),\ s),\ x)$

THEOREM: length-non-equal
$(\mathrm{length}\,(x) < \mathrm{length}\,(y)) \rightarrow ((x = y) = \mathbf{f})$

THEOREM: first-branch-ok-for-succard
$(\mathrm{subseq}\,(s,\ \textit{big-s}) \wedge \text{node-p}\,(\textit{big-s}) \wedge (s \neq \textit{big-s}))$
$\rightarrow$ $((\text{first-branch}\,(s,\ \textit{big-s}) \in \mathbf{N})$
$\wedge$ $(0 < \text{first-branch}\,(s,\ \textit{big-s}))$
$\wedge$ $(\mathrm{succard}\,(s) \not< \text{first-branch}\,(s,\ \textit{big-s})))$

THEOREM: all-big-h-lessp-linear
$((0 < i) \wedge (\mathrm{succard}\,(s) \not< i))$
$\rightarrow$ $(\text{big-h}\,(\mathrm{cons}\,(i,\ s)) < \text{all-big-h}\,(s,\ \mathrm{succard}\,(s)))$

EVENT: Disable all-big-h-lessp.

THEOREM: inf-implies-inf-next
$(\text{node-p}\,(s) \wedge \mathrm{inf}\,(s)) \rightarrow \mathrm{inf}\,(\mathrm{next}\,(s,\ \mathrm{succard}\,(s)))$

DEFINITION:
$\mathrm{k}\,(n)$
$=$ **if** $n \simeq 0$ **then nil**
else $\mathrm{next}\,(\mathrm{k}\,(n-1),\ \mathrm{succard}\,(\mathrm{k}\,(n-1)))$ **endif**

THEOREM: subseq-nil
$\mathrm{subseq}\,(\mathbf{nil},\ x) = \mathrm{plistp}\,(x)$

THEOREM: node-p-implies-plistp
$\text{node-p}\,(s) \rightarrow \mathrm{plistp}\,(s)$

THEOREM: inf-nil
$\mathrm{inf}\,(\mathbf{nil})$

EVENT: Disable node-p-implies-plistp.

THEOREM: konig-tree-lemma-1
$\mathrm{inf}\,(\mathrm{k}\,(n))$

THEOREM: length-next
$\inf(x) \rightarrow (\text{length}(\text{next}(s,\ n)) = (1 + \text{length}(s)))$

THEOREM: konig-tree-lemma-2
$\text{length}(\text{k}(n)) = \text{fix}(n)$

THEOREM: subseq-next
$\text{subseq}(s1,\ s2) \rightarrow \text{subseq}(s1,\ \text{next}(s2,\ n))$

THEOREM: konig-tree-lemma-3
$(j \not< i) \rightarrow \text{subseq}(\text{k}(i),\ \text{k}(j))$

THEOREM: konig-tree-lemma
$\text{node-p}(\text{k}(n))$
$\wedge \quad ((j \not< i) \rightarrow \text{subseq}(\text{k}(i),\ \text{k}(j)))$
$\wedge \quad (\text{length}(\text{k}(n)) = \text{fix}(n))$

```
;; or, if one prefers:
```

THEOREM: konig-tree-lemma-again
$(n \in \mathbf{N})$
$\rightarrow \quad (\text{node-p}(\text{k}(n))$
$\wedge \quad ((j \not< i) \rightarrow \text{subseq}(\text{k}(i),\ \text{k}(j)))$
$\wedge \quad (\text{length}(\text{k}(n)) = n))$

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