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;; Matt Kaufmann

;; Ramsey Theorem (infinite version) events supporting ``An Extension
;; of the Boyer-Moore Theorem Prover to Support First-Order
;; Quantification,'' to appear in JAR (1992?).  The DEFN-SK events
;; have been replaced by DCLs and ADD-AXIOMs, as shown.

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;; Ramsey Theorem Events List
;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
```

EVENT: Start with the initial **nqthm** theory.

CONSERVATIVE AXIOM: p-num-intro

$$
\begin{array}{l}
((x \in \mathbf{N}) \land (y \in \mathbf{N})) \\
\rightarrow \quad ((0 < \text{p-num}\,(x,\ y)) \\
\qquad \land \quad (\textsc{bound} \not< \text{p-num}\,(x,\ y)) \\
\qquad \land \quad (\text{p-num}\,(x,\ y) = \text{p-num}\,(y,\ x)))
\end{array}
$$

Simultaneously, we introduce the new function symbols *p-num* and *bound*.
EVENT: Disable p-num-intro.

DEFINITION: $\mathrm{p}\,(x,\ y) = \text{p-num}\,(\mathrm{fix}\,(x), \mathrm{fix}\,(y))$

THEOREM: p-intro
$(\mathtt{0} < \mathrm{p}\,(x,\ y)) \land (\textsc{bound} \not< \mathrm{p}\,(x,\ y)) \land (\mathrm{p}\,(x,\ y) = \mathrm{p}\,(y,\ x))$

EVENT: Disable p.

DEFINITION:
prehom-seq-1 $(a,\ x)$
$=$ **if** listp (x)
then (p (caar (x), a) $=$ cdar (x)) $\land$ prehom-seq-1 $(a$, cdr $(x))$
else t endif

DEFINITION:
prehom-seq (x)
$=$ **if** listp (x)
then if listp (cdr (x))
then (car (cadr (x)) $<$ car (car (x)))
$\land$ prehom-seq-1 (caar (x), cdr (x))
$\land$ prehom-seq (cdr (x))
else t endif
else t endif

```
#| The original DEFN-SK event here was processed as follows:

>(defn-sk extensible (s)
  ;; s is a list of pairs (i . c), and extensible means that there
  ;; are infinitely many a for which prehom-seq-1(a,s) holds.
  (forall above
          (exists next
                  (and (lessp above next)
                       (prehom-seq-1 next s)))))


Adding the Skolem axiom:
      (AND (IMPLIES (AND (LESSP (ABOVE S) NEXT)
                         (PREHOM-SEQ-1 NEXT S))
                    (EXTENSIBLE S))
           (IMPLIES (NOT (AND (LESSP ABOVE (NEXT ABOVE S))
                              (PREHOM-SEQ-1 (NEXT ABOVE S) S)))
                    (NOT (EXTENSIBLE S)))).
```

```
     As this is a DEFN-SK we can conclude that:
      (OR (TRUEP (EXTENSIBLE S))
          (FALSEP (EXTENSIBLE S)))
is a theorem.


[ 0.3 0.0 0.0 ]
EXTENSIBLE

>

|#
```

EVENT: Introduce the function symbol *above* of one argument.

EVENT: Introduce the function symbol *next* of 2 arguments.

EVENT: Introduce the function symbol *extensible* of one argument.

AXIOM: extensible-intro

$$(((\text{above}\,(s) < \mathit{next}) \land \text{prehom-seq-1}\,(\mathit{next},\ s)) \rightarrow \text{extensible}\,(s))$$
$$\land \quad ((\neg\ ((\mathit{above} < \text{next}\,(\mathit{above},\ s)) \land \text{prehom-seq-1}\,(\text{next}\,(\mathit{above},\ s),\ s))) \rightarrow \quad (\neg\ \text{extensible}\,(s)))$$

AXIOM: extensible-boolean

$$\text{truep}\,(\text{extensible}\,(s)) \lor \text{falsep}\,(\text{extensible}\,(s))$$

```
;; The following two lemmas are for the benefit of the proof-checker
;; macro-command SK*.
```

THEOREM: extensible-suff

$$((\text{above}\,(s) < \mathit{next}) \land \text{prehom-seq-1}\,(\mathit{next},\ s)) \rightarrow \text{extensible}\,(s)$$

THEOREM: extensible-necc

$$(\neg\ ((\mathit{above} < \text{next}\,(\mathit{above},\ s)) \land \text{prehom-seq-1}\,(\text{next}\,(\mathit{above},\ s),\ s)))$$
$$\rightarrow \quad (\neg\ \text{extensible}\,(s))$$

DEFINITION:

$$\text{above-all-aux}\,(a,\ s,\ n)$$
$$= \quad \textbf{if } n \simeq 0 \textbf{ then } 0$$
$$\quad \textbf{else } \text{above}\,(\text{cons}\,(\text{cons}\,(a,\ n),\ s)) + \text{above-all-aux}\,(a,\ s,\ n-1) \textbf{ endif}$$

DEFINITION: above-all (a, s) = above-all-aux (a, s, BOUND)

THEOREM: lessp-above-all-aux
$(0 < n) \rightarrow (\text{above-all-aux}\,(a, s, n) \not< \text{above}\,(\text{cons}\,(\text{cons}\,(a, n), s)))$

THEOREM: above-all-aux-monotone
$(\mathit{bound} \not< n) \rightarrow (\text{above-all-aux}\,(a, s, \mathit{bound}) \not< \text{above-all-aux}\,(a, s, n))$

THEOREM: lessp-above-all-bound
$((0 < n) \land (\text{BOUND} \not< n))$
$\rightarrow \quad (\text{above-all}\,(a, s) \not< \text{above}\,(\text{cons}\,(\text{cons}\,(a, n), s)))$

DEFINITION: next-element (a, s) = next (above-all (a, s), s)

DEFINITION: next-color (a, s) = p $(a$, next-element $(a, s))$

THEOREM: numberp-p
$\text{p}\,(x, y) \in \mathbf{N}$

THEOREM: next-color-bound
$(0 < \text{next-color}\,(a, s)) \land (\text{BOUND} \not< \text{next-color}\,(a, s))$

EVENT: Disable above-all.

```
;; first of two goals for extensible-cons
```

THEOREM: lessp-next-element
extensible (s)
$\rightarrow \quad (\text{above}\,(\text{cons}\,(\text{cons}\,(a, \text{next-color}\,(a, s)), s)) < \text{next-element}\,(a, s))$

THEOREM: prehom-seq-1-next-element
extensible $(s) \rightarrow$ prehom-seq-1 (next-element (a, s), s)

EVENT: Disable next-element.

```
;; second of two goals for extensible-cons
```

THEOREM: prehom-seq-1-next-element-cons
extensible (s)
$\rightarrow \quad$ prehom-seq-1 (next-element (a, s), cons (cons $(a$, next-color $(a, s))$, s))

EVENT: Disable next-color.

THEOREM: extensible-cons
extensible $(s) \rightarrow$ extensible (cons (cons $(a$, next-color $(a, s))$, s))

DEFINITION:
next-pair (s) = cons (next (caar (s), s), next-color (next (caar (s), s), s))

THEOREM: next-pair-extends
extensible $(s) \rightarrow$ extensible (cons (next-pair (s), s))

DEFINITION: ramsey-seq-p (s) = (extensible (s) $\wedge$ prehom-seq (s))

THEOREM: extensible-next-property
extensible $(s) \rightarrow ((a <$ next $(a, s)) \wedge$ prehom-seq-1 (next (a, s), s))

THEOREM: ramsey-seq-p-extends
ramsey-seq-p $(s) \rightarrow$ ramsey-seq-p (cons (next-pair (s), s))

THEOREM: extensible-nlistp
$(\neg$ listp $(s)) \rightarrow$ extensible (s)

THEOREM: ramsey-seq-p-nlistp
$(s \simeq$ **nil**$) \rightarrow$ ramsey-seq-p (s)

EVENT: Disable ramsey-seq-p.

EVENT: Disable next-pair.

DEFINITION:
ramsey-seq (n)
= **if** $n \simeq 0$ **then nil**
else cons (next-pair (ramsey-seq $(n - 1)$), ramsey-seq $(n - 1)$) **endif**

THEOREM: ramsey-seq-has-ramsey-seq-p
ramsey-seq-p (ramsey-seq (n))

```
;; Now we want to cut down this prehomogenous sequence to one that's
;; homogeneous.  First let's define the flag.

#| The original DEFN-SK event here was processed as follows:

>(defn-sk good-color-p (c)
  ;; says that arbitrarily large elements of ramsey-seq agree with c
  (forall big
          (exists good-c-index
                  (and (lessp big good-c-index)
                       (equal c (cdr (car (ramsey-seq good-c-index)))))))
```

```
Adding the Skolem axiom:
      (AND
          (IMPLIES (AND (LESSP (BIG C) GOOD-C-INDEX)
                        (EQUAL C
                               (CDAR (RAMSEY-SEQ GOOD-C-INDEX))))
                   (GOOD-COLOR-P C))
          (IMPLIES (NOT (AND (LESSP BIG (GOOD-C-INDEX BIG C))
                             (EQUAL C
                                    (CDAR (RAMSEY-SEQ (GOOD-C-INDEX BIG C))))))
                   (NOT (GOOD-COLOR-P C)))).


     As this is a DEFN-SK we can conclude that:
      (OR (TRUEP (GOOD-COLOR-P C))
          (FALSEP (GOOD-COLOR-P C)))
is a theorem.


[ 0.2 0.0 0.0 ]
GOOD-COLOR-P

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```

EVENT: Introduce the function symbol *big* of one argument.

EVENT: Introduce the function symbol *good-c-index* of 2 arguments.

EVENT: Introduce the function symbol *good-color-p* of one argument.

AXIOM: good-color-p-intro

$$
\begin{array}{l}
(((\mathrm{big}\,(c) < \mathit{good\text{-}c\text{-}index}) \land (c = \mathrm{cdar}\,(\mathrm{ramsey\text{-}seq}\,(\mathit{good\text{-}c\text{-}index}))))\\
\quad \rightarrow \quad \mathrm{good\text{-}color\text{-}p}\,(c))\\
\land \quad ((\neg\,((\mathit{big} < \mathrm{good\text{-}c\text{-}index}\,(\mathit{big},\ c))\\
\qquad\qquad \land \quad (c = \mathrm{cdar}\,(\mathrm{ramsey\text{-}seq}\,(\mathrm{good\text{-}c\text{-}index}\,(\mathit{big},\ c))))))\\
\qquad \rightarrow \quad (\neg\ \mathrm{good\text{-}color\text{-}p}\,(c)))
\end{array}
$$

AXIOM: good-color-p-boolean

$$\mathrm{truep}\,(\mathrm{good\text{-}color\text{-}p}\,(c)) \lor \mathrm{falsep}\,(\mathrm{good\text{-}color\text{-}p}\,(c))$$

```
;; The following two lemmas are for the benefit of the proof-checker
```

```
;; macro-command SK*.
```

THEOREM: good-color-p-suff
$((\text{big}(c) < \textit{good-c-index}) \wedge (c = \text{cdar}(\text{ramsey-seq}(\textit{good-c-index}))))$
$\rightarrow \quad \text{good-color-p}(c)$

THEOREM: good-color-p-necc
$(\neg ((\textit{big} < \text{good-c-index}(\textit{big}, c))$
$\quad \wedge \quad (c = \text{cdar}(\text{ramsey-seq}(\text{good-c-index}(\textit{big}, c))))))$
$\rightarrow \quad (\neg \text{good-color-p}(c))$

DEFINITION:
good-c-index-wit$(\textit{bound})$
$= \quad$ **if** $\textit{bound} \simeq \mathtt{0}$ **then** $\mathtt{1}$
else $\text{big}(\textit{bound}) + \text{good-c-index-wit}(\textit{bound} - 1)$ **endif**

THEOREM: good-c-index-wit-positive
$\mathtt{0} < \text{good-c-index-wit}(\textit{bound})$

THEOREM: lessp-big-good-c-index
$((\mathtt{0} < c) \wedge (\textit{bound} \not< c))$
$\rightarrow \quad ((\text{big}(c) < \text{good-c-index-wit}(\textit{bound})) = \mathbf{t})$

EVENT: Disable good-c-index-wit.

DEFINITION: $\quad \text{COLOR} = \text{cdar}(\text{ramsey-seq}(\text{good-c-index-wit}(\text{BOUND})))$

THEOREM: ramsey-seq-has-colors-in-bounds
$(\mathtt{0} < n)$
$\rightarrow \quad ((\mathtt{0} < \text{cdar}(\text{ramsey-seq}(n))) \wedge (\text{BOUND} \not< \text{cdar}(\text{ramsey-seq}(n))))$

THEOREM: color-in-bounds
$(\mathtt{0} < \text{COLOR}) \wedge (\text{BOUND} \not< \text{COLOR})$

THEOREM: color-is-good
good-color-p (COLOR)

EVENT: Disable color.

EVENT: Disable *1*color.

DEFINITION:
ramsey-index(n)
$= \quad$ **if** $n \simeq \mathtt{0}$ **then** good-c-index$(\mathtt{0}, \text{COLOR})$
else good-c-index$(\text{ramsey-index}(n - 1), \text{COLOR})$ **endif**

THEOREM: color-properties
$(big < \text{good-c-index}(big,\ \text{COLOR}))$
$\wedge\quad (\text{cdr}(\text{car}(\text{ramsey-seq}(\text{good-c-index}(big,\ \text{COLOR})))) = \text{COLOR})$

THEOREM: ramsey-index-increasing
$(i < j) \rightarrow (\text{ramsey-index}(i) < \text{ramsey-index}(j))$

DEFINITION: $\text{ramsey}(n) = \text{car}(\text{car}(\text{ramsey-seq}(\text{ramsey-index}(n))))$

```
; Next goal:
;(prove-lemma ramsey-increasing (rewrite)
;  (implies (lessp i j)
;           (lessp (ramsey i) (ramsey j))))
```

THEOREM: good-c-index-numberp
$\text{good-c-index}(big,\ \text{COLOR}) \in \mathbf{N}$

THEOREM: ramsey-index-numberp
$\text{ramsey-index}(n) \in \mathbf{N}$

THEOREM: car-next-pair
$\text{car}(\text{next-pair}(s)) = \text{next}(\text{caar}(s),\ s)$

THEOREM: ramsey-seq-extensible
$\text{extensible}(\text{ramsey-seq}(n))$

THEOREM: next-not-zerop
$\text{extensible}(s) \rightarrow ((\text{next}(a,\ s) \in \mathbf{N}) \wedge (\text{next}(a,\ s) \neq \mathtt{0}))$

THEOREM: ramsey-seq-increasing
$(i < j) \rightarrow ((\text{caar}(\text{ramsey-seq}(i)) < \text{caar}(\text{ramsey-seq}(j))) = \mathbf{t})$

THEOREM: ramsey-index-increasing-rewrite
$(i < j) \rightarrow ((\text{ramsey-index}(i) < \text{ramsey-index}(j)) = \mathbf{t})$

EVENT: Disable ramsey-index-increasing.

THEOREM: ramsey-increasing
$(i < j) \rightarrow (\text{ramsey}(i) < \text{ramsey}(j))$

```
;; Now we want to show that ramsey is homogeneous for (color):

;(prove-lemma ramsey-homogeneous (rewrite)
;  (implies (lessp i j)
;           (iff (p (ramsey i) (ramsey j))
;                (color))))
```

DEFINITION:
restn $(n,\ l)$
$=$ **if** listp (l)
then if $n \simeq \mathtt{0}$ **then** l
else restn $(n - 1,\ \text{cdr}\,(l))$ **endif**
else l **endif**

```
;; We've already proved (ramsey-seq-p (ramseq-seq i)).  So, in order
;; to prove the key lemma ramsey-seq-prehom below, we'll use this
;; together with an appropriate fact about restn and prehom seqs.
```

THEOREM: ramsey-seq-restn-length
restn $(n,\ \text{ramsey-seq}\,(n)) = $ **nil**

THEOREM: prehom-seq-ramsey
prehom-seq (ramsey-seq (n))

DEFINITION:
length (l)
$=$ **if** listp (l) **then** $1 + \text{length}\,(\text{cdr}\,(l))$
else $\mathtt{0}$ **endif**

THEOREM: prehom-seq-1-restn
(prehom-seq-1 $(a,\ s) \land (x < \text{length}\,(s)))$
$\rightarrow$ (p (caar (restn $(x,\ s)),\ a) = $ cdar (restn $(x,\ s)))$

THEOREM: prehom-seq-restn
(prehom-seq $(s) \land (\mathtt{0} < x) \land (x < \text{length}\,(s)))$
$\rightarrow$ (p (caar (restn $(x,\ s))$, caar $(s)) = $ cdar (restn $(x,\ s)))$

THEOREM: ramsey-seq-plus
restn $(x,\ \text{ramsey-seq}\,(x + y)) = \text{ramsey-seq}\,(y)$

THEOREM: plus-comm
$(x + y) = (y + x)$

THEOREM: ramsey-seq-plus-commuted
restn $(x,\ \text{ramsey-seq}\,(y + x)) = \text{ramsey-seq}\,(y)$

THEOREM: length-ramsey-seq
length (ramsey-seq $(n)) = \text{fix}\,(n)$

```
;; The lemmas from here to RAMSEY-SEQ-PREHOM were to eliminate the
;; proof-checker hints from that lemma.
```

THEOREM: plus-difference-elim
$((j \in \mathbf{N}) \wedge (j \not< i)) \rightarrow ((i + (j - i)) = j)$

THEOREM: restn-difference-ramsey-seq
$((0 < i) \wedge (i < j))$
$\rightarrow \quad (\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j)) = \text{ramsey-seq}\,(i))$

THEOREM: prehom-seq-restn-commuted
$(\text{prehom-seq}\,(s) \wedge (0 < x) \wedge (x < \text{length}\,(s)))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(s),\ \text{caar}\,(\text{restn}\,(x,\ s))) = \text{cdar}\,(\text{restn}\,(x,\ s)))$

THEOREM: ramsey-seq-prehom-lemma
$((0 < i) \wedge (i < j))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j))),\ \text{caar}\,(\text{ramsey-seq}\,(j)))$
$\qquad = \quad \text{cdar}\,(\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j))))$

THEOREM: ramsey-seq-prehom
$((0 < i) \wedge (i < j))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{ramsey-seq}\,(i)),\ \text{caar}\,(\text{ramsey-seq}\,(j))) = \text{cdar}\,(\text{ramsey-seq}\,(i)))$

THEOREM: cdar-ramseq-seq-ramsey-index
$\text{cdar}\,(\text{ramsey-seq}\,(\text{ramsey-index}\,(n))) = \text{COLOR}$

THEOREM: lessp-good-c-index
$(big < \text{good-c-index}\,(big,\ \text{COLOR})) = \mathbf{t}$

EVENT: Disable color-properties.

THEOREM: good-c-index-non-zero
$(0 < \text{good-c-index}\,(big,\ \text{COLOR})) = \mathbf{t}$

THEOREM: ramsey-index-positive
$0 < \text{ramsey-index}\,(n)$

THEOREM: ramsey-seq-hom-lessp
$(i < j) \rightarrow (\text{p}\,(\text{ramsey}\,(i),\ \text{ramsey}\,(j)) = \text{COLOR})$

THEOREM: p-num-is-p
$((x \in \mathbf{N}) \wedge (y \in \mathbf{N})) \rightarrow (\text{p-num}\,(x,\ y) = \text{p}\,(x,\ y))$

THEOREM: numberp-ramsey
$\text{caar}\,(\text{ramsey-seq}\,(\text{ramsey-index}\,(n))) \in \mathbf{N}$

THEOREM: ramsey-seq-hom
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (i \neq j))$
$\rightarrow \quad (\text{p-num}\,(\text{ramsey}\,(i),\ \text{ramsey}\,(j)) = \text{COLOR})$

```
;; The above, together with what was already proved, i.e.

;; (prove-lemma ramsey-increasing (rewrite)
;;  (implies (lessp i j)
;;           (lessp (ramsey i) (ramsey j))))

;; and the fact that p-num was arbitrary (and there are no add-axioms)
;; finishes the job.
```

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