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|#

```
;; A solution to the Gilbreath card trick challenge.
;; Matt Kaufmann, 10/92.
```

```
;; The proof splits into two halves. The lemma main-1 handles the
;; case in which we do not make the final adjustment of rotating one
;; card. The lemma main-2 handles the other case. We glue these
;; together in the final theorem, main.
```

EVENT: Start with the initial **nqthm** theory.

DEFINITION:

```
length(x)
=  if listp(x) then 1 + length(cdr(x))
    else 0 endif
```

```
;; The following definition takes an arbitrary 'oracle', which says
;; whether the next card in the shuffle comes from the left pile or
;; the right pile. See below for the definition of shuffle-top,
;; which makes the 'move one card' adjustment when necessary.
```

DEFINITION:

```
shuffle(left, right, oracle)  
=  if left  $\simeq$  nil then right  
    elseif right  $\simeq$  nil then left  
    elseif car(oracle)  
        then cons(car(left), shuffle(cdr(left), right, cdr(oracle)))  
        else cons(car(right), shuffle(left, cdr(right), cdr(oracle))) endif
```

```
;; To be really arbitrary, we postulate a color function that takes  
;; two values (which might as well be booleans), using the  
;; conservative CONSTRAIN principle to make this postulation.
```

CONSERVATIVE AXIOM: color-intro

```
truep(color(x))  $\vee$  falsep(color(x))
```

Simultaneously, we introduce the new function symbol *color*.

DEFINITION: same-color(*x*, *y*) = (color(*x*) \leftrightarrow color(*y*))

DEFINITION:

```
altp(pile)  
=  if listp(pile)  
    then if listp(cdr(pile))  
        then if same-color(car(pile), cadr(pile)) then f  
            else altp(cdr(pile)) endif  
        else t endif  
    else t endif
```

DEFINITION:

```
last(x)  
=  if listp(x)  $\wedge$  listp(cdr(x)) then last(cdr(x))  
    else car(x) endif
```

DEFINITION:

```
butlast(x)  
=  if listp(x)  $\wedge$  listp(cdr(x)) then cons(car(x), butlast(cdr(x)))  
    else nil endif
```

DEFINITION:

```
shuffle-top(left, right, oracle)  
=  let shuf be shuffle(left, right, oracle)  
    in  
    if  $\neg$  same-color(last(left), last(right)) then shuf  
    else cons(last(shuf), butlast(shuf)) endif endlet
```

DEFINITION:

even-length-p-rec (x)
= **if** listp (x) **then** \neg even-length-p-rec (cdr (x))
 else t endif

THEOREM: even-length-p-rec-rewrite

(listp (x) \wedge altp (x))
 \rightarrow (even-length-p-rec (x) = (\neg same-color (car (x), last (x))))

THEOREM: even-length-p-rec-append

even-length-p-rec (append (x , y))
= (even-length-p-rec (x) \leftrightarrow even-length-p-rec (y))

;; A conditional rewrite rule with hypothesis
;; (and (listp x) (listp y)) would probably suffice for most of the
;; proof, but I believe that this version is needed somewhere late in
;; the proof.

THEOREM: altp-append

altp (append (x , y))
= (altp (x)
 \wedge altp (y)
 \wedge ((listp (x) \wedge listp (y)) \rightarrow (\neg same-color (last (x), car (y))))))

;; Probably the following lemma isn't needed till the end, but let's
;; make sure that we can prove it. Once we know this to be true,
;; we'll use it implicitly by adding in the hypothesis
;; (not (same-color (car left) (car right))) to main-1 (and similarly
;; for main-2).

THEOREM: last-same-color-iff-first-same-color

(listp ($left$)
 \wedge listp ($right$)
 \wedge altp (append ($left$, $right$))
 \wedge even-length-p-rec (append ($left$, $right$)))
 \rightarrow (same-color (last ($left$), last ($right$))
 \leftrightarrow same-color (car ($left$), car ($right$)))

;; Here is the induction scheme we use for main-1, and in fact also
;; for main-2 (actually main-2-lemma).

DEFINITION:

```

main-1-induction (left, right, oracle)
=  if (left  $\simeq$  nil)  $\vee$  (right  $\simeq$  nil) then t
    elseif car (oracle)
        then if cadr (oracle)
            then main-1-induction (cddr (left), right, cddr (oracle))
            else main-1-induction (cdr (left), cdr (right), cddr (oracle)) endif
        elseif cadr (oracle)
            then main-1-induction (cdr (left), cdr (right), cddr (oracle))
            else main-1-induction (left, cddr (right), cddr (oracle)) endif

```

DEFINITION:

```

alt2-p (x)
=  if listp (x)
    then if listp (cdr (x))
        then ( $\neg$  same-color (car (x), cadr (x)))  $\wedge$  alt2-p (cddr (x))
        else f endif
    else t endif

```

THEOREM: altp-implies-alt2-p

$\text{altp} (x) \rightarrow (\text{alt2-p} (x) = \text{even-length-p-rec} (x))$

THEOREM: last-append

$\text{listp} (y) \rightarrow (\text{last} (\text{append} (x, y)) = \text{last} (y))$

;; Now we may prove a version of the first half.

THEOREM: main-1

```

(listp (left)
 $\wedge$  listp (right)
 $\wedge$  even-length-p-rec (append (left, right))
 $\wedge$  altp (left)
 $\wedge$  altp (right)
 $\wedge$  ( $\neg$  same-color (car (left), car (right)))
 $\wedge$  ( $\neg$  same-color (last (left), last (right))))
 $\rightarrow$  alt2-p (shuffle (left, right, oracle))

```

;; For the other half, we modify the notion of alt2-p (calling the
;; result by the weird name alt3-p), except that we expect an odd
;; number of cards. Our strategy is to first prove that in the second
;; case, the CDR of the shuffle has this alt3-p property
;; (main-2-lemma). Then we can show that when we move the final card
;; of the shuffle to the top, the result is an alt2-p. The lemma
;; alt3-p-to-alt2-p below lets us do that little adjustment, once we
;; know (by the lemma shuffle-preserves-reds-equal-blacks) that the

```
;; shuffle has the property that the numbers of red and black cards in
;; it are the same.
```

DEFINITION:

```
alt3-p(x)
=  if listp(x)
    then if listp(cdr(x))
         then (¬ same-color(car(x), cadr(x))) ∧ alt3-p(cddr(x))
         else t endif
    else f endif
```

```
;; The following lemma is analogous to one proved for main-1; then the
;; proof of main-2-lemma goes through.
```

THEOREM: altp-implies-alt3-p

```
altp(x) → (alt3-p(x) = (¬ even-length-p-rec(x)))
```

THEOREM: main-2-lemma

```
(listp(left)
 ∧ listp(right)
 ∧ even-length-p-rec(append(left, right))
 ∧ altp(left)
 ∧ altp(right)
 ∧ same-color(car(left), car(right))
 ∧ same-color(last(left), last(right)))
 → alt3-p(cdr(shuffle(left, right, oracle)))
```

DEFINITION:

```
count-color(color, x)
=  if listp(x)
    then if color = color(car(x)) then 1 + count-color(color, cdr(x))
         else count-color(color, cdr(x)) endif
    else 0 endif
```

DEFINITION:

```
reds-equal-blacks(x) = (count-color(t, x) = count-color(f, x))
```

THEOREM: alt-implies-reds-equal-blacks

```
altp(x)
 → if even-length-p-rec(x) then count-color(t, x) = count-color(f, x)
    elseif color(car(x)) then count-color(t, x)
                                = (1 + count-color(f, x))
    else (1 + count-color(t, x)) = count-color(f, x) endif
```

THEOREM: count-color-shuffle
 $\text{count-color}(color, \text{shuffle}(x, y, oracle))$
 $= \text{count-color}(color, x) + \text{count-color}(color, y)$

THEOREM: shuffle-preserves-reds-equal-blacks
 $(\text{altp}(\text{append}(x, y)) \wedge \text{even-length-p-rec}(\text{append}(x, y)))$
 $\rightarrow \text{reds-equal-blacks}(\text{shuffle}(x, y, oracle))$

THEOREM: alt3-p-to-alt2-p
 $(\text{reds-equal-blacks}(\text{cons}(a, x)) \wedge \text{alt3-p}(x))$
 $\rightarrow \text{alt2-p}(\text{cons}(\text{last}(x), \text{butlast}(\text{cons}(a, x))))$

THEOREM: shuffle-cdr
 $(\text{listp}(x) \wedge \text{listp}(y))$
 $\rightarrow (\text{listp}(\text{shuffle}(x, y, oracle)) \wedge \text{listp}(\text{cdr}(\text{shuffle}(x, y, oracle))))$

THEOREM: main-2
 $(\text{listp}(left)$
 $\wedge \text{listp}(right)$
 $\wedge \text{even-length-p-rec}(\text{append}(left, right))$
 $\wedge \text{altp}(left)$
 $\wedge \text{altp}(right)$
 $\wedge \text{altp}(\text{append}(left, right))$
 $\wedge \text{same-color}(\text{car}(left), \text{car}(right))$
 $\wedge \text{same-color}(\text{last}(left), \text{last}(right)))$
 $\rightarrow \text{alt2-p}(\text{shuffle-top}(left, right, oracle))$

;; The following three events are there simply to show that our
 ;; definition of ‘‘even length’’ is honest.

DEFINITION: $\text{even-length-p}(x) = ((\text{length}(x) \bmod 2) = 0)$

THEOREM: remainder-2-add1
 $((1 + x) \bmod 2 = 0) = (x \bmod 2 \neq 0)$

THEOREM: even-length-p-is-even-length-p-rec
 $\text{even-length-p}(x) = \text{even-length-p-rec}(x)$

THEOREM: main
 $(\text{listp}(left)$
 $\wedge \text{listp}(right)$
 $\wedge \text{even-length-p}(\text{append}(left, right))$
 $\wedge \text{altp}(\text{append}(left, right)))$
 $\rightarrow \text{alt2-p}(\text{shuffle-top}(left, right, oracle))$

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