EVENT: Start with the initial nqthm theory.

; load basic definitions and lemmas

; The following is one of the examples I'm using in my course here
; to illustrate nqthm. In particular, note that the representation
; of a pair of numbers by an ordinal, as described on p. 42, is more
; complicated than it has to be.

Ken

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nqthm contains induction on \( \epsilon_0 \), so it's stronger than pure primitive recursive arithmetic. Presumably, it can prove Con(PA).

This file -- a simple example -- use recursion on pairs to define the Ackermann function, which grows faster than any primitive recursive function; see Aho-Hopcroft-Ullman, "Data Structures and Algorithms", p. 189

Definition: \( \text{rep}(i, j) = \text{cons}(1 + i, j) \)

Definition:
\[
\text{lex2}(i_1, j_1, i_2, j_2) = ((i_1 < i_2) \lor ((i_1 = i_2) \land (j_1 < j_2)))
\]

Theorem: rep-respects-lex
\[
((i_1 \in \textbf{N}) \land (i_2 \in \textbf{N}) \land (j_1 \in \textbf{N}) \land (j_2 \in \textbf{N})) \implies \text{lex2}(i_1, j_1, i_2, j_2) = \text{ord-lessp}(\text{rep}(i_1, j_1), \text{rep}(i_2, j_2))
\]

Definition:
\[
\text{ack}(x, y) = \begin{cases} 1 & \text{if } x \approx 0 \\ \text{if } y \approx 0 \\ \text{then if } x = 1 \text{ then 2} \\ \text{else } x + 2 \text{ endif} \\ \text{else } \text{ack}(\text{ack}(x - 1, y), y - 1) \text{ endif} \end{cases}
\]

; hint
; "fix" = "cast to numberp"

Theorem: ack-is-positive
\( \text{ack}(x, y) \approx 0 \) = \( f \)

Theorem: ack-of-1
\( x \not\approx 0 \) \( \rightarrow \) (\( \text{ack}(x, 1) = (x \ast 2) \))
Definition:
\[ \text{expt2}(x) = \begin{cases} 
1 & \text{if } x \simeq 0 \\
\text{expt2}(x - 1) \times 2 & \text{else}
\end{cases} \]

Theorem: ack-of-2-aux1
\[ (x \not\simeq 0) \rightarrow (\text{ack}(x, 2) = \text{ack}(\text{ack}(x - 1, 2), 1)) \]

Theorem: ack-of-2-aux2
\[ (x \not\simeq 0) \rightarrow (\text{ack}(x, 2) = (\text{ack}(x - 1, 2) \times 2)) \]

Theorem: ack-of-2
\[ \text{ack}(x, 2) = \text{expt2}(x) \]

; \text{ack}(x, 3) = 2^2^2^\ldots^2 \text{ (stack of x 2's, } ^\text{ assoc to right)}
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