```
#|

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|#

;            Proof of the Correctness of a GCD Program
```

EVENT: Start with the library "mc20-2" using the compiled version.

```
#|

The following LISP program computes the greatest common divisor of two
integers a and b. We, here, investigate the machine code of this
program generated by a widely used C compiler, and verify the
correctness of the code.

; computes the greatest common divisor by Euclid's algorithm.
(proclaim '(function fixnum-gcd (fixnum fixnum) fixnum))

(defun fixnum-gcd (x y)
  (declare (fixnum x y))
  (if (not (= x 0))
      (if (= y 0)
  (return-from fixnum-gcd x)
```

```
(if (> x y)
    (fixnum-gcd (the fixnum (rem x y)) y)
  (fixnum-gcd x (the fixnum (rem y x)))))
    (return-from fixnum-gcd y)))
```

Here is the MC68020 assembly code of the above GCD program. The code is generated by AKCL.

```
0x14 <LI1>: linkw fp,#0
0x18 <LI1+4>: moveml d2-d3,sp@-
0x1c <LI1+8>: movel fp@(8),d3
0x20 <LI1+12>: movel fp@(12),d2
0x24 <LI1+16>: tstl d3
0x26 <LI1+18>: beq 0x44 <LI1+48>
0x28 <LI1+20>: tstl d2
0x2a <LI1+22>: bne 0x30 <LI1+28>
0x2c <LI1+24>: movel d3,d0
0x2e <LI1+26>: bra 0x46 <LI1+50>
0x30 <LI1+28>: cmpl d3,d2
0x32 <LI1+30>: bge 0x3c <LI1+40>
0x34 <LI1+32>: divsll d2,d0,d3
0x38 <LI1+36>: movel d0,d3
0x3a <LI1+38>: bra 0x24 <LI1+16>
0x3c <LI1+40>: divsll d3,d0,d2
0x40 <LI1+44>: movel d0,d2
0x42 <LI1+46>: bra 0x24 <LI1+16>
0x44 <LI1+48>: movel d2,d0
0x46 <LI1+50>: moveml fp@(-8),d2-d3
0x4c <LI1+56>: unlk fp
0x4e <LI1+58>: rts
```

The machine code of the above program is:

```
<LI1>:     0x4e56  0x0000  0x48e7  0x3000  0x262e  0x0008  0x242e  0x000c
<LI1+16>:  0x4a83  0x671c  0x4a82  0x6604  0x2003  0x6016  0xb483  0x6c08
<LI1+32>:  0x4c42  0x3800  0x2600  0x60e8  0x4c43  0x2800  0x2400  0x60e0
<LI1+48>:  0x2002  0x4cee  0x000c  0xfff8  0x4e5e  0x4e75
```

```
  '(78 86 0 0 72 231 48 0
    38 46 0 8 36 46 0 12
    74 131 103 28 74 130 102 4
    32 3 96 22 180 131 108 8
    76 66 56 0 38 0 96 232
    76 67 40 0 36 0 96 224
```

```
    32 2 76 238 0 12 255 248
    78 94 78 117)
|#

; now we start the correctness proof of this GCD program, defined by
; (gcd-code).
```

DEFINITION:
GCD-CODE
= `'(78 86 0 0 72 231 48 0 38 46 0 8 36 46 0 12 74 131`
`103 28 74 130 102 4 32 3 96 22 180 131 108 8 76 66`
`56 0 38 0 96 232 76 67 40 0 36 0 96 224 32 2 76 238`
`0 12 255 248 78 94 78 117)`

```
; the functional description of the program (gcd-code).
```

DEFINITION:
gcd $(a,\ b)$
= **if** $a \simeq 0$ **then** fix $(b)$
**elseif** $b \simeq 0$ **then** $a$
**elseif** $b < a$ **then** gcd $(a$ **mod** $b,\ b)$
**else** gcd $(a,\ b$ **mod** $a)$ **endif**

DEFINITION:
gcd-t1 $(a,\ b)$
= **if** $a \simeq 0$ **then** 6
**elseif** $b \simeq 0$ **then** 9
**elseif** $b < a$ **then** splus $(9,$ gcd-t1 $(a$ **mod** $b,\ b))$
**else** splus $(9,$ gcd-t1 $(a,\ b$ **mod** $a))$ **endif**

DEFINITION: gcd-t $(a,\ b)$ = splus $(4,$ gcd-t1 $(a,\ b))$

DEFINITION:
gcd-induct $(s,\ a,\ b)$
= **if** $(a \simeq 0) \vee (b \simeq 0)$ **then t**
**elseif** $b < a$ **then** gcd-induct (stepn $(s,\ 9),\ a$ **mod** $b,\ b)$
**else** gcd-induct (stepn $(s,\ 9),\ a,\ b$ **mod** $a)$ **endif**

DEFINITION:
gcd-statep $(s,\ a,\ b)$
= ((mc-status $(s)$ = `'running`)
$\wedge$ evenp (mc-pc $(s)$)
$\wedge$ rom-addrp (mc-pc $(s)$, mc-mem $(s)$, 60)
$\wedge$ mcode-addrp (mc-pc $(s)$, mc-mem $(s)$, GCD-CODE)
$\wedge$ ram-addrp (sub (32, 12, read-sp $(s)$), mc-mem $(s)$, 24)

$\wedge$ $(a =$ iread-mem (add (32, read-sp ($s$), 4), mc-mem ($s$), 4))
$\wedge$ $(b =$ iread-mem (add (32, read-sp ($s$), 8), mc-mem ($s$), 4))
$\wedge$ $(a \in \mathbf{N})$
$\wedge$ $(b \in \mathbf{N}))$

DEFINITION:
gcd-s0p ($s$, $a$, $b$)
= ((mc-status ($s$) = 'running)
$\wedge$ evenp (mc-pc ($s$))
$\wedge$ rom-addrp (sub (32, 16, mc-pc ($s$)), mc-mem ($s$), 60)
$\wedge$ mcode-addrp (sub (32, 16, mc-pc ($s$)), mc-mem ($s$), GCD-CODE)
$\wedge$ ram-addrp (sub (32, 8, read-an (32, 6, $s$)), mc-mem ($s$), 24)
$\wedge$ $(a =$ iread-dn (32, 3, $s$))
$\wedge$ $(b =$ iread-dn (32, 2, $s$))
$\wedge$ $(a \in \mathbf{N})$
$\wedge$ $(b \in \mathbf{N}))$

```
; s --> s0.
```

THEOREM: gcd-s-s0
gcd-statep ($s$, $a$, $b$)
$\rightarrow$ (gcd-s0p (stepn ($s$, 4), $a$, $b$)
$\wedge$ (linked-rts-addr (stepn ($s$, 4)) = rts-addr ($s$))
$\wedge$ (linked-a6 (stepn ($s$, 4)) = read-an (32, 6, $s$))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn ($s$, 4)))
= sub (32, 4, read-sp ($s$)))
$\wedge$ (movem-saved (stepn ($s$, 4), 4, 8, 2)
= readm-rn (32, '(2 3), mc-rfile ($s$))))

THEOREM: gcd-s-s0-rfile
(gcd-statep ($s$, $a$, $b$) $\wedge$ d4-7a2-5p ($rn$))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, 4)))
= read-rn ($oplen$, $rn$, mc-rfile ($s$)))

THEOREM: gcd-s-s0-mem
(gcd-statep ($s$, $a$, $b$) $\wedge$ disjoint ($x$, $l$, sub (32, 12, read-sp ($s$)), 24))
$\rightarrow$ (read-mem ($x$, mc-mem (stepn ($s$, 4)), $l$) = read-mem ($x$, mc-mem ($s$), $l$))

```
; s0 --> exit.
; base case: s0 --> exit.
```

THEOREM: gcd-s0-sn-base-1
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \simeq$ 0))
$\rightarrow$ ((mc-status (stepn ($s$, 6)) = 'running)
$\wedge$ (mc-pc (stepn ($s$, 6)) = linked-rts-addr ($s$))

$\wedge$ (iread-dn (32, 0, stepn ($s$, 6)) = fix ($b$))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn ($s$, 6))) = linked-a6 ($s$))
$\wedge$ (read-rn (32, 15, mc-rfile (stepn ($s$, 6)))
= add (32, read-an (32, 6, $s$), 8))
$\wedge$ (read-mem ($x$, mc-mem (stepn ($s$, 6)), $l$)
= read-mem ($x$, mc-mem ($s$), $l$)))

THEOREM: gcd-s0-sn-base-rfile-1
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \simeq$ 0) $\wedge$ d2-7a2-5p ($rn$) $\wedge$ ($oplen \leq$ 32))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, 6)))
= **if** d4-7a2-5p ($rn$) **then** read-rn ($oplen$, $rn$, mc-rfile ($s$))
**else** get-vlst ($oplen$, 0, $rn$, '(2 3), movem-saved ($s$, 4, 8, 2)) **endif**)

THEOREM: gcd-s0-sn-base-2
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \simeq$ 0))
$\rightarrow$ ((mc-status (stepn ($s$, 9)) = 'running)
$\wedge$ (mc-pc (stepn ($s$, 9)) = linked-rts-addr ($s$))
$\wedge$ (iread-dn (32, 0, stepn ($s$, 9)) = fix ($a$))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn ($s$, 9))) = linked-a6 ($s$))
$\wedge$ (read-rn (32, 15, mc-rfile (stepn ($s$, 9)))
= add (32, read-an (32, 6, $s$), 8))
$\wedge$ (read-mem ($x$, mc-mem (stepn ($s$, 9)), $l$)
= read-mem ($x$, mc-mem ($s$), $l$)))

THEOREM: gcd-s0-sn-base-rfile-2
(gcd-s0p ($s$, $a$, $b$)
$\wedge$ ($a \not\simeq$ 0)
$\wedge$ ($b \simeq$ 0)
$\wedge$ d2-7a2-5p ($rn$)
$\wedge$ ($oplen \leq$ 32))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, 9)))
= **if** d4-7a2-5p ($rn$) **then** read-rn ($oplen$, $rn$, mc-rfile ($s$))
**else** get-vlst ($oplen$, 0, $rn$, '(2 3), movem-saved ($s$, 4, 8, 2)) **endif**)

```
; induction case:  s0 --> s0.
```

EVENT: Enable integerp.

EVENT: Enable iremainder.

THEOREM: gcd-s0-s0-1
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \not\simeq$ 0) $\wedge$ ($b < a$))
$\rightarrow$ (gcd-s0p (stepn ($s$, 9), $a$ **mod** $b$, $b$)
$\wedge$ (read-rn ($oplen$, 14, mc-rfile (stepn ($s$, 9)))

$=$ read-rn ($oplen$, 14, mc-rfile ($s$)))
$\wedge$ (linked-a6 (stepn ($s$, 9)) = linked-a6 ($s$))
$\wedge$ (linked-rts-addr (stepn ($s$, 9)) = linked-rts-addr ($s$))
$\wedge$ (movem-saved (stepn ($s$, 9), 4, 8, 2) = movem-saved ($s$, 4, 8, 2))
$\wedge$ (read-mem ($x$, mc-mem (stepn ($s$, 9)), $l$)
$=$ read-mem ($x$, mc-mem ($s$), $l$)))

THEOREM: gcd-s0-s0-2
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \not\simeq$ 0) $\wedge$ ($b \not< a$))
$\rightarrow$ (gcd-s0p (stepn ($s$, 9), $a$, $b$ **mod** $a$)
$\wedge$ (read-rn ($oplen$, 14, mc-rfile (stepn ($s$, 9)))
$=$ read-rn ($oplen$, 14, mc-rfile ($s$)))
$\wedge$ (linked-a6 (stepn ($s$, 9)) = linked-a6 ($s$))
$\wedge$ (linked-rts-addr (stepn ($s$, 9)) = linked-rts-addr ($s$))
$\wedge$ (movem-saved (stepn ($s$, 9), 4, 8, 2) = movem-saved ($s$, 4, 8, 2))
$\wedge$ (read-mem ($x$, mc-mem (stepn ($s$, 9)), $l$)
$=$ read-mem ($x$, mc-mem ($s$), $l$)))

THEOREM: gcd-s0-s0-rfile-1
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \not\simeq$ 0) $\wedge$ ($b < a$) $\wedge$ d4-7a2-5p ($rn$))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, 9)))
$=$ read-rn ($oplen$, $rn$, mc-rfile ($s$)))

THEOREM: gcd-s0-s0-rfile-2
(gcd-s0p ($s$, $a$, $b$) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \not\simeq$ 0) $\wedge$ ($b \not< a$) $\wedge$ d4-7a2-5p ($rn$))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, 9)))
$=$ read-rn ($oplen$, $rn$, mc-rfile ($s$)))

```
; put together.
```

THEOREM: gcd-s0-sn
gcd-s0p ($s$, $a$, $b$)
$\rightarrow$ ((mc-status (stepn ($s$, gcd-t1 ($a$, $b$))) = 'running)
$\wedge$ (mc-pc (stepn ($s$, gcd-t1 ($a$, $b$))) = linked-rts-addr ($s$))
$\wedge$ (iread-dn (32, 0, stepn ($s$, gcd-t1 ($a$, $b$))) = gcd ($a$, $b$))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn ($s$, gcd-t1 ($a$, $b$))))
$=$ linked-a6 ($s$))
$\wedge$ (read-rn (32, 15, mc-rfile (stepn ($s$, gcd-t1 ($a$, $b$))))
$=$ add (32, read-an (32, 6, $s$), 8))
$\wedge$ (read-mem ($x$, mc-mem (stepn ($s$, gcd-t1 ($a$, $b$))), $k$)
$=$ read-mem ($x$, mc-mem ($s$), $k$)))

THEOREM: gcd-s0-sn-rfile
(gcd-s0p ($s$, $a$, $b$) $\wedge$ d2-7a2-5p ($rn$) $\wedge$ ($oplen \leq$ 32))
$\rightarrow$ (read-rn ($oplen$, $rn$, mc-rfile (stepn ($s$, gcd-t1 ($a$, $b$))))
$=$ **if** d4-7a2-5p ($rn$) **then** read-rn ($oplen$, $rn$, mc-rfile ($s$))
**else** get-vlst ($oplen$, 0, $rn$, '(2 3), movem-saved ($s$, 4, 8, 2)) **endif**)

```
; the correctness of gcd.
```

THEOREM: gcd-correctness
gcd-statep $(s,\ a,\ b)$
$\rightarrow$ ((mc-status (stepn $(s,$ gcd-t $(a,\ b)))) =$ `'running`)
$\wedge$ (mc-pc (stepn $(s,$ gcd-t $(a,\ b))) =$ rts-addr $(s)$)
$\wedge$ (iread-dn (`32`, `0`, stepn $(s,$ gcd-t $(a,\ b))) =$ gcd $(a,\ b)$)
$\wedge$ (read-rn (`32`, `14`, mc-rfile (stepn $(s,$ gcd-t $(a,\ b))))
= read-rn (`32`, `14`, mc-rfile $(s)$))
$\wedge$ (read-rn (`32`, `15`, mc-rfile (stepn $(s,$ gcd-t $(a,\ b))))
= add (`32`, read-an (`32`, `7`, $s$), `4`)))

THEOREM: gcd-rfile
(gcd-statep $(s,\ a,\ b) \wedge$ d2-7a2-5p $(rn) \wedge (oplen \leq$ `32`$)$)
$\rightarrow$ (read-rn $(oplen,\ rn,$ mc-rfile (stepn $(s,$ gcd-t $(a,\ b))))$
= read-rn $(oplen,\ rn,$ mc-rfile $(s)))$

THEOREM: gcd-mem
(gcd-statep $(s,\ a,\ b) \wedge$ disjoint $(x,\ k,$ sub (`32`, `12`, read-sp $(s)$), `24`))
$\rightarrow$ (read-mem $(x,$ mc-mem (stepn $(s,$ gcd-t $(a,\ b))),\ k)$
= read-mem $(x,$ mc-mem $(s),\ k))$

EVENT: Disable gcd-t.

```
; tp prove that (gcd a b) does compute the greatest common divisor, we need
; to prove the following two theorems:
;     1. (gcd a b) divides a and (gcd a b) divides b.
;        i.e. (gcd a b) is a common divisor of a and b.
;     2. any divisor of a and b is no greater than (gcd a b).
```

THEOREM: remainder-remainder
$((b$ **mod** $c) =$ `0`$) \rightarrow (((a$ **mod** $b)$ **mod** $c) = (a$ **mod** $c))$

THEOREM: gcd-is-cd
$((a$ **mod** gcd $(a,\ b)) =$ `0`$) \wedge ((b$ **mod** gcd $(a,\ b)) =$ `0`

THEOREM: gcd-the-greatest
$((a \not\simeq$ `0`$) \wedge (b \not\simeq$ `0`$) \wedge ((a$ **mod** $x) =$ `0`$) \wedge ((b$ **mod** $x) =$ `0`$))$
$\rightarrow$ (gcd $(a,\ b) \not< x)$

```
; a simple timing analysis.
```

THEOREM: lessp-times-lessp
$((a < b) \wedge (c \not\simeq$ `0`$)) \rightarrow (a < (b * c))$

THEOREM: remainder-shrink-half
$((b \leq a) \wedge (b \not\simeq \mathtt{0})) \rightarrow ((a \div \mathtt{2}) \not< (a \textbf{ mod } b))$

THEOREM: gcd-t-shrink-1
$((b \leq a) \wedge (b \not\simeq \mathtt{0})) \rightarrow ((\log(\mathtt{2}, a) - 1) \not< \log(\mathtt{2}, a \textbf{ mod } b))$

THEOREM: gcd-t-shrink-2
$((b \leq a) \wedge (b \not\simeq \mathtt{0}) \wedge (a \neq \mathtt{1}))$
$\rightarrow \quad ((\mathtt{9} * (x + \log(\mathtt{2}, a)))$
$\not< \quad (\mathtt{9} + (\mathtt{9} * (x + \log(\mathtt{2}, a \textbf{ mod } b)))))$

DEFINITION:
gcd-t2$(a, b)$
$=$ **if** $a \simeq \mathtt{0}$ **then** $\mathtt{6}$
**elseif** $b \simeq \mathtt{0}$ **then** $\mathtt{9}$
**elseif** $b < a$ **then** $\mathtt{9} +$ gcd-t2$(a \textbf{ mod } b, b)$
**elseif** $a < b$ **then** $\mathtt{9} +$ gcd-t2$(a, b \textbf{ mod } a)$
**else** $\mathtt{18}$ **endif**

THEOREM: gcd-t2-ub
gcd-t2$(a, b) \leq (\mathtt{18} + (\mathtt{9} * (\log(\mathtt{2}, a) + \log(\mathtt{2}, b))))$

THEOREM: gcd-t1-t2
gcd-t1$(a, b) =$ gcd-t2$(a, b)$

THEOREM: gcd-t-ub
gcd-t$(a, b) \leq (\mathtt{22} + (\mathtt{9} * (\log(\mathtt{2}, a) + \log(\mathtt{2}, b))))$

THEOREM: gcd-t-ubound-la
$((a \leq a1) \wedge (b \leq b1))$
$\rightarrow \quad ((\mathtt{22} + (\mathtt{9} * (\log(\mathtt{2}, a) + \log(\mathtt{2}, b))))$
$\leq \quad (\mathtt{22} + (\mathtt{9} * (\log(\mathtt{2}, a1) + \log(\mathtt{2}, b1)))))$

THEOREM: gcd-t-ubound
$((a < \exp(\mathtt{2}, \mathtt{32})) \wedge (b < \exp(\mathtt{2}, \mathtt{32}))) \rightarrow (\text{gcd-t}(a, b) \leq \mathtt{598})$

# Index