```
#|

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|#

;            Proof of the Correctness of a GCD Program
```

EVENT: Start with the library "mc20-2" using the compiled version.

```
#|

The following C program computes the greatest common divisor of two
integers a and b. We, here, investigate the machine code of this
program generated by a widely used C compiler, and verify the
correctness of the code.

/* computes the greatest common divisor by Euclid's algorithm */
gcd(int a, int b)
{
  while (a != 0){
    if (b == 0) return (a);
    if (a > b)
      a = a % b;
    else b = b % a;
```

```
  };
  return (b);
}
```

Here is the MC68020 assembly code of the above GCD program. The code is generated by "gcc -O".

```
0x22c6 <gcd>:           linkw a6,#0
0x22ca <gcd+4>:         moveml d2-d3,sp@-
0x22ce <gcd+8>:         movel a6@(8),d2
0x22d2 <gcd+12>:        movel a6@(12),d3
0x22d6 <gcd+16>:        tstl d2
0x22d8 <gcd+18>:        beq 0x22f6 <gcd+48>
0x22da <gcd+20>:        tstl d3
0x22dc <gcd+22>:        bne 0x22e2 <gcd+28>
0x22de <gcd+24>:        movel d2,d0
0x22e0 <gcd+26>:        bra 0x22f8 <gcd+50>
0x22e2 <gcd+28>:        cmpl d2,d3
0x22e4 <gcd+30>:        bge 0x22ee <gcd+40>
0x22e6 <gcd+32>:        divsll d3,d0,d2
0x22ea <gcd+36>:        movel d0,d2
0x22ec <gcd+38>:        bra 0x22d6 <gcd+16>
0x22ee <gcd+40>:        divsll d2,d0,d3
0x22f2 <gcd+44>:        movel d0,d3
0x22f4 <gcd+46>:        bra 0x22d6 <gcd+16>
0x22f6 <gcd+48>:        movel d3,d0
0x22f8 <gcd+50>:        moveml a6@(-8),d2-d3
0x22fe <gcd+56>:        unlk a6
0x2300 <gcd+58>:        rts
```

The machine code of the above program is:

```
<gcd>:       0x4e56  0x0000  0x48e7  0x3000  0x242e  0x0008  0x262e  0x000c
<gcd+16>:    0x4a82  0x671c  0x4a83  0x6604  0x2002  0x6016  0xb682  0x6c08
<gcd+32>:    0x4c43  0x2800  0x2400  0x60e8  0x4c42  0x3800  0x2600  0x60e0
<gcd+48>:    0x2003  0x4cee  0x000c  0xfff8  0x4e5e  0x4e75
```

```
'(78      86      0       0       72      231     48      0
  36      46      0       8       38      46      0       12
  74      130     103     28      74      131     102     4
  32      2       96      22      182     130     108     8
  76      67      40      0       36      0       96      232
  76      66      56      0       38      0       96      224
  32      3       76      238     0       12      255     248
```

```
  78      94      78      117)
|#

; now we start the correctness proof of this GCD program, defined by
; (gcd-code).
```

DEFINITION:
GCD-CODE
= '(78 86 0 0 72 231 48 0 36 46 0 8 38 46 0 12 74 130
103 28 74 131 102 4 32 2 96 22 182 130 108 8 76 67
40 0 36 0 96 232 76 66 56 0 38 0 96 224 32 3 76 238
0 12 255 248 78 94 78 117)

CONSERVATIVE AXIOM: gcd-load
gcd-loadp (s)
= (evenp (GCD-ADDR)
$\wedge$ (GCD-ADDR $\in$ **N**)
$\wedge$ nat-rangep (GCD-ADDR, 32)
$\wedge$ rom-addrp (GCD-ADDR, mc-mem (s), 60)
$\wedge$ mcode-addrp (GCD-ADDR, mc-mem (s), GCD-CODE))

Simultaneously, we introduce the new function symbols *gcd-loadp* and *gcd-addr*.

THEOREM: stepn-gcd-loadp
gcd-loadp (stepn (s, n)) = gcd-loadp (s)

```
; the functional description of the program (gcd-code).
```

DEFINITION:
gcd (a, b)
= **if** $a \simeq 0$ **then** fix (b)
elseif $b \simeq 0$ **then** a
elseif $b < a$ **then** gcd (a **mod** b, b)
else gcd (a, b **mod** a) **endif**

```
; the clock function.
```

DEFINITION:
gcd-t1 (a, b)
= **if** $a \simeq 0$ **then** 6
elseif $b \simeq 0$ **then** 9
elseif $b < a$ **then** splus (9, gcd-t1 (a **mod** b, b))
else splus (9, gcd-t1 (a, b **mod** a)) **endif**

DEFINITION: gcd-t (a, b) = splus (4, gcd-t1 (a, b))

```
; an induction hint.
```

DEFINITION:
gcd-induct (s, a, b)
= **if** ($a \simeq 0$) $\vee$ ($b \simeq 0$) **then t**
elseif $b < a$ **then** gcd-induct (stepn (s, 9), a **mod** b, b)
else gcd-induct (stepn (s, 9), a, b **mod** a) **endif**

```
; the initial state.
```

DEFINITION:
gcd-statep (s, a, b)
= ((mc-status (s) = 'running)
$\wedge$ evenp (mc-pc (s))
$\wedge$ rom-addrp (mc-pc (s), mc-mem (s), 60)
$\wedge$ mcode-addrp (mc-pc (s), mc-mem (s), GCD-CODE)
$\wedge$ ram-addrp (sub (32, 12, read-sp (s)), mc-mem (s), 24)
$\wedge$ (a = iread-mem (add (32, read-sp (s), 4), mc-mem (s), 4))
$\wedge$ (b = iread-mem (add (32, read-sp (s), 8), mc-mem (s), 4))
$\wedge$ ($a \in \mathbf{N}$)
$\wedge$ ($b \in \mathbf{N}$))

```
; an intermediate state.
```

DEFINITION:
gcd-s0p (s, a, b)
= ((mc-status (s) = 'running)
$\wedge$ evenp (mc-pc (s))
$\wedge$ rom-addrp (sub (32, 16, mc-pc (s)), mc-mem (s), 60)
$\wedge$ mcode-addrp (sub (32, 16, mc-pc (s)), mc-mem (s), GCD-CODE)
$\wedge$ ram-addrp (sub (32, 8, read-an (32, 6, s)), mc-mem (s), 24)
$\wedge$ (a = iread-dn (32, 2, s))
$\wedge$ (b = iread-dn (32, 3, s))
$\wedge$ ($a \in \mathbf{N}$)
$\wedge$ ($b \in \mathbf{N}$))

```
; s --> s0.
```

THEOREM: gcd-s-s0
gcd-statep (s, a, b)
$\rightarrow$ (gcd-s0p (stepn (s, 4), a, b)
$\wedge$ (linked-rts-addr (stepn (s, 4)) = rts-addr (s))
$\wedge$ (linked-a6 (stepn (s, 4)) = read-an (32, 6, s))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn (s, 4)))
= sub (32, 4, read-sp (s)))
$\wedge$ (movem-saved (stepn (s, 4), 4, 8, 2)
= readm-rn (32, '(2 3), mc-rfile (s))))

THEOREM: gcd-s-s0-rfile
(gcd-statep (s, a, b) $\wedge$ d4-7a2-5p (rn))
$\rightarrow$ (read-rn ($oplen$, rn, mc-rfile (stepn (s, 4)))
$=$ read-rn ($oplen$, rn, mc-rfile (s)))

THEOREM: gcd-s-s0-mem
(gcd-statep (s, a, b) $\wedge$ disjoint (x, l, sub (32, 12, read-sp (s)), 24))
$\rightarrow$ (read-mem (x, mc-mem (stepn (s, 4)), l) = read-mem (x, mc-mem (s), l))

```
; s0 --> exit.
; base case: s0 --> exit.
```

THEOREM: gcd-s0-sn-base-1
(gcd-s0p (s, a, b) $\wedge$ ($a \simeq$ 0))
$\rightarrow$ ((mc-status (stepn (s, 6)) = 'running)
$\wedge$ (mc-pc (stepn (s, 6)) = linked-rts-addr (s))
$\wedge$ (iread-dn (32, 0, stepn (s, 6)) = fix (b))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn (s, 6))) = linked-a6 (s))
$\wedge$ (read-rn (32, 15, mc-rfile (stepn (s, 6)))
$=$ add (32, read-an (32, 6, s), 8))
$\wedge$ (read-mem (x, mc-mem (stepn (s, 6)), l)
$=$ read-mem (x, mc-mem (s), l)))

THEOREM: gcd-s0-sn-base-rfile-1
(gcd-s0p (s, a, b) $\wedge$ ($a \simeq$ 0) $\wedge$ d2-7a2-5p (rn) $\wedge$ ($oplen \leq$ 32))
$\rightarrow$ (read-rn ($oplen$, rn, mc-rfile (stepn (s, 6)))
$=$ **if** d4-7a2-5p (rn) **then** read-rn ($oplen$, rn, mc-rfile (s))
else get-vlst ($oplen$, 0, rn, '(2 3), movem-saved (s, 4, 8, 2)) **endif**)

THEOREM: gcd-s0-sn-base-2
(gcd-s0p (s, a, b) $\wedge$ ($a \not\simeq$ 0) $\wedge$ ($b \simeq$ 0))
$\rightarrow$ ((mc-status (stepn (s, 9)) = 'running)
$\wedge$ (mc-pc (stepn (s, 9)) = linked-rts-addr (s))
$\wedge$ (iread-dn (32, 0, stepn (s, 9)) = fix (a))
$\wedge$ (read-rn (32, 14, mc-rfile (stepn (s, 9))) = linked-a6 (s))
$\wedge$ (read-rn (32, 15, mc-rfile (stepn (s, 9)))
$=$ add (32, read-an (32, 6, s), 8))
$\wedge$ (read-mem (x, mc-mem (stepn (s, 9)), l)
$=$ read-mem (x, mc-mem (s), l)))

THEOREM: gcd-s0-sn-base-rfile-2
(gcd-s0p (s, a, b)
$\wedge$ ($a \not\simeq$ 0)
$\wedge$ ($b \simeq$ 0)
$\wedge$ d2-7a2-5p (rn)

$\wedge$ $(oplen \leq 32)$)
$\rightarrow$ (read-rn ($oplen$, rn, mc-rfile (stepn (s, 9)))
= **if** d4-7a2-5p (rn) **then** read-rn ($oplen$, rn, mc-rfile (s))
else get-vlst ($oplen$, 0, rn, '(2 3), movem-saved (s, 4, 8, 2)) **endif**)

```
; induction case:  s0 --> s0.
```

EVENT: Enable integerp.

EVENT: Enable iremainder.

THEOREM: gcd-s0-s0-1
(gcd-s0p (s, a, b) $\wedge$ ($a \not\simeq 0$) $\wedge$ ($b \not\simeq 0$) $\wedge$ ($b < a$))
$\rightarrow$ (gcd-s0p (stepn (s, 9), a **mod** b, b)
$\wedge$ (read-rn ($oplen$, 14, mc-rfile (stepn (s, 9)))
= read-rn ($oplen$, 14, mc-rfile (s)))
$\wedge$ (linked-a6 (stepn (s, 9)) = linked-a6 (s))
$\wedge$ (linked-rts-addr (stepn (s, 9)) = linked-rts-addr (s))
$\wedge$ (movem-saved (stepn (s, 9), 4, 8, 2) = movem-saved (s, 4, 8, 2))
$\wedge$ (read-mem (x, mc-mem (stepn (s, 9)), l)
= read-mem (x, mc-mem (s), l)))

THEOREM: gcd-s0-s0-2
(gcd-s0p (s, a, b) $\wedge$ ($a \not\simeq 0$) $\wedge$ ($b \not\simeq 0$) $\wedge$ ($b \not< a$))
$\rightarrow$ (gcd-s0p (stepn (s, 9), a, b **mod** a)
$\wedge$ (read-rn ($oplen$, 14, mc-rfile (stepn (s, 9)))
= read-rn ($oplen$, 14, mc-rfile (s)))
$\wedge$ (linked-a6 (stepn (s, 9)) = linked-a6 (s))
$\wedge$ (linked-rts-addr (stepn (s, 9)) = linked-rts-addr (s))
$\wedge$ (movem-saved (stepn (s, 9), 4, 8, 2) = movem-saved (s, 4, 8, 2))
$\wedge$ (read-mem (x, mc-mem (stepn (s, 9)), l)
= read-mem (x, mc-mem (s), l)))

THEOREM: gcd-s0-s0-rfile-1
(gcd-s0p (s, a, b) $\wedge$ ($a \not\simeq 0$) $\wedge$ ($b \not\simeq 0$) $\wedge$ ($b < a$) $\wedge$ d4-7a2-5p (rn))
$\rightarrow$ (read-rn ($oplen$, rn, mc-rfile (stepn (s, 9)))
= read-rn ($oplen$, rn, mc-rfile (s)))

THEOREM: gcd-s0-s0-rfile-2
(gcd-s0p (s, a, b) $\wedge$ ($a \not\simeq 0$) $\wedge$ ($b \not\simeq 0$) $\wedge$ ($b \not< a$) $\wedge$ d4-7a2-5p (rn))
$\rightarrow$ (read-rn ($oplen$, rn, mc-rfile (stepn (s, 9)))
= read-rn ($oplen$, rn, mc-rfile (s)))

```
; put together.
```

THEOREM: gcd-s0-sn
let *sn* **be** stepn (*s*, gcd-t1 (*a*, *b*))
in
gcd-s0p (*s*, *a*, *b*)
→ ((mc-status (*sn*) = `'running`)
∧ (mc-pc (*sn*) = linked-rts-addr (*s*))
∧ (iread-dn (32, 0, *sn*) = gcd (*a*, *b*))
∧ (read-rn (32, 14, mc-rfile (*sn*)) = linked-a6 (*s*))
∧ (read-rn (32, 15, mc-rfile (*sn*))
= add (32, read-an (32, 6, *s*), 8))
∧ (read-mem (*x*, mc-mem (*sn*), *k*) = read-mem (*x*, mc-mem (*s*), *k*))) **endlet**

THEOREM: gcd-s0-sn-rfile
(gcd-s0p (*s*, *a*, *b*) ∧ d2-7a2-5p (*rn*) ∧ (*oplen* ≤ 32))
→ (read-rn (*oplen*, *rn*, mc-rfile (stepn (*s*, gcd-t1 (*a*, *b*))))
= **if** d4-7a2-5p (*rn*) **then** read-rn (*oplen*, *rn*, mc-rfile (*s*))
else get-vlst (*oplen*, 0, *rn*, `'(2 3)`, movem-saved (*s*, 4, 8, 2)) **endif**)

```
; the correctness of gcd.
```

THEOREM: gcd-correctness
let *sn* **be** stepn (*s*, gcd-t (*a*, *b*))
in
gcd-statep (*s*, *a*, *b*)
→ ((mc-status (*sn*) = `'running`)
∧ (mc-pc (*sn*) = rts-addr (*s*))
∧ (read-rn (32, 14, mc-rfile (*sn*))
= read-rn (32, 14, mc-rfile (*s*)))
∧ (read-rn (32, 15, mc-rfile (*sn*))
= add (32, read-an (32, 7, *s*), 4))
∧ ((d2-7a2-5p (*rn*) ∧ (*oplen* ≤ 32))
→ (read-rn (*oplen*, *rn*, mc-rfile (*sn*))
= read-rn (*oplen*, *rn*, mc-rfile (*s*))))
∧ (disjoint (*x*, *k*, sub (32, 12, read-sp (*s*)), 24)
→ (read-mem (*x*, mc-mem (*sn*), *k*)
= read-mem (*x*, mc-mem (*s*), *k*)))
∧ (iread-dn (32, 0, *sn*) = gcd (*a*, *b*))) **endlet**

EVENT: Disable gcd-t.

```
; to prove that (gcd a b) does compute the greatest common divisor, we need
; to prove the following two theorems:
;     1. (gcd a b) divides a and (gcd a b) divides b.
;        i.e. (gcd a b) is a common divisor of a and b.
;     2. any divisor of a and b is no greater than (gcd a b).
```

THEOREM: remainder-remainder
$((b \textbf{ mod } c) = 0) \rightarrow (((a \textbf{ mod } b) \textbf{ mod } c) = (a \textbf{ mod } c))$

THEOREM: gcd-is-cd
$((a \textbf{ mod } \text{gcd}\,(a,\ b)) = 0) \wedge ((b \textbf{ mod } \text{gcd}\,(a,\ b)) = 0)$

THEOREM: gcd-the-greatest
$((a \not\simeq 0) \wedge (b \not\simeq 0) \wedge ((a \textbf{ mod } x) = 0) \wedge ((b \textbf{ mod } x) = 0))$
$\rightarrow \quad (\text{gcd}\,(a,\ b) \not< x)$

```
; a simple timing analysis.
```

THEOREM: lessp-times-lessp
$((a < b) \wedge (c \not\simeq 0)) \rightarrow (a < (b * c))$

THEOREM: remainder-shrink-half
$((b \leq a) \wedge (b \not\simeq 0)) \rightarrow ((a \div 2) \not< (a \textbf{ mod } b))$

THEOREM: gcd-t-shrink-1
$((b \leq a) \wedge (b \not\simeq 0)) \rightarrow ((\log(2,\ a) - 1) \not< \log(2,\ a \textbf{ mod } b))$

THEOREM: gcd-t-shrink-2
$((b \leq a) \wedge (b \not\simeq 0) \wedge (a \neq 1))$
$\rightarrow \quad ((9 * (x + \log(2,\ a)))$
$\not< \quad (9 + (9 * (x + \log(2,\ a \textbf{ mod } b)))))$

DEFINITION:
gcd-t2 $(a,\ b)$
= **if** $a \simeq 0$ **then** 6
elseif $b \simeq 0$ **then** 9
elseif $b < a$ **then** 9 + gcd-t2 $(a \textbf{ mod } b,\ b)$
elseif $a < b$ **then** 9 + gcd-t2 $(a,\ b \textbf{ mod } a)$
else 18 **endif**

THEOREM: gcd-t2-ub
$\text{gcd-t2}\,(a,\ b) \leq (18 + (9 * (\log(2,\ a) + \log(2,\ b))))$

THEOREM: gcd-t1-t2
$\text{gcd-t1}\,(a,\ b) = \text{gcd-t2}\,(a,\ b)$

THEOREM: gcd-t-ub
$\text{gcd-t}\,(a,\ b) \leq (22 + (9 * (\log(2,\ a) + \log(2,\ b))))$

THEOREM: gcd-t-ubound
$((a < \exp(2,\ 31)) \wedge (b < \exp(2,\ 31))) \rightarrow (\text{gcd-t}\,(a,\ b) \leq 580)$

EVENT: Make the library "gcd" and compile it.

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