```
; Works as of 4/06/87 20:14:00
```

EVENT: Start with the initial **nqthm** theory.

DEFINITION:
min $(x,\ y)$
= **if** $x < y$ **then** x
else y **endif**

DEFINITION:
nth $(i,\ l)$
= **if** $i \simeq \texttt{0}$ **then** car (l)
else nth $(i - 1,\ \text{cdr}\,(l))$ **endif**

DEFINITION:
put $(n,\ val,\ l)$
= **if** $n \simeq \texttt{0}$ **then** cons $(val,\ \text{cdr}\,(l))$
else cons $(\text{car}\,(l),\ \text{put}\,(n - 1,\ val,\ \text{cdr}\,(l)))$ **endif**

THEOREM: nth-of-put
$\text{nth}\,(i,\ \text{put}\,(i,\ v,\ l)) = v$

THEOREM: nth-of-put-neq
$((i \neq j) \wedge (i \in \mathbf{N}) \wedge (j \in \mathbf{N})) \rightarrow (\text{nth}\,(i,\ \text{put}\,(j,\ v,\ l)) = \text{nth}\,(i,\ l))$

DEFINITION:
$\text{access}\,(matrix,\ i,\ j) = (\neg\ (\neg\ \text{nth}\,(j,\ \text{nth}\,(i,\ matrix))))$

EVENT: Disable access.

DEFINITION:
length (lst)
= **if** listp (lst) **then** $1 + \text{length}\,(\text{cdr}\,(lst))$
else 0 **endif**

DEFINITION: $\text{col}\,(matrix) = \text{length}\,(\text{car}\,(matrix))$

EVENT: Disable col.

DEFINITION:
zeroes (n)
= **if** $n \simeq \texttt{0}$ **then nil**
else cons $(\texttt{0},\ \text{zeroes}\,(n - 1))$ **endif**

DEFINITION:
segmentp $(i,\ j,\ \mathit{left},\ \mathit{matrix})$
$=$ **if** $\mathit{left} \simeq 0$ **then t**
else access $(\mathit{matrix},\ i,\ j) \land$ segmentp $(i,\ j-1,\ \mathit{left}-1,\ \mathit{matrix})$ **endif**

THEOREM: segmentp-access
$((i \in \mathbf{N}) \land (j \in \mathbf{N}) \land$ segmentp $(i,\ j,\ \mathit{left},\ \mathit{matrix}) \land (k < \mathit{left}))$
$\rightarrow$ access $(\mathit{matrix},\ i,\ j-k)$

DEFINITION:
rectanglep $(i,\ j,\ \mathit{left},\ \mathit{up},\ \mathit{matrix})$
$=$ **if** $(\mathit{left} \simeq 0) \lor (\mathit{up} \simeq 0)$ **then t**
else rectanglep $(i-1,\ j,\ \mathit{left},\ \mathit{up}-1,\ \mathit{matrix})$
$\land$ segmentp $(i,\ j,\ \mathit{left},\ \mathit{matrix})$ **endif**

THEOREM: rectanglep-access
$((i \in \mathbf{N})$
$\land$ $(j \in \mathbf{N})$
$\land$ $(\mathit{up} \in \mathbf{N})$
$\land$ $(\mathit{up} \neq 0)$
$\land$ rectanglep $(i,\ j,\ \mathit{left},\ \mathit{up},\ \mathit{matrix})$
$\land$ $(k < \mathit{left}))$
$\rightarrow$ access $(\mathit{matrix},\ i,\ j-k)$

DEFINITION:
tsquarep $(i,\ j,\ \mathit{side},\ \mathit{matrix})$
$=$ (rectanglep $(i,\ j,\ \mathit{side},\ \mathit{side},\ \mathit{matrix})$
$\land$ $((1+i) \not< \mathit{side})$
$\land$ $((1+j) \not< \mathit{side}))$

EVENT: Disable tsquarep.

THEOREM: tsquarep-0
tsquarep $(i,\ j,\ 0,\ \mathit{matrix})$

THEOREM: segmentp-goes-down
$((m \not< n) \land$ segmentp $(i,\ j,\ m,\ \mathit{matrix})) \rightarrow$ segmentp $(i,\ j,\ n,\ \mathit{matrix})$

THEOREM: rectanglep-goes-down
$((\mathit{m1} \not< \mathit{n1}) \land (\mathit{m2} \not< \mathit{n2}) \land$ rectanglep $(i,\ j,\ \mathit{m1},\ \mathit{m2},\ \mathit{matrix}))$
$\rightarrow$ rectanglep $(i,\ j,\ \mathit{n1},\ \mathit{n2},\ \mathit{matrix})$

THEOREM: tsquarep-goes-down
$((m \not< n) \land$ tsquarep $(i,\ j,\ m,\ \mathit{matrix})) \rightarrow$ tsquarep $(i,\ j,\ n,\ \mathit{matrix})$

DEFINITION:
ok-c-before $(i,\ j,\ c,\ \mathit{matrix})$
$=$ **if** $j \simeq 0$ **then t**
else tsquarep $(i,\ j-1,\ \text{nth}\,(j-1,\ c),\ \mathit{matrix})$
$\wedge$ ($\neg$ tsquarep $(i,\ j-1,\ 1+\text{nth}\,(j-1,\ c),\ \mathit{matrix})$)
$\wedge$ ok-c-before $(i,\ j-1,\ c,\ \mathit{matrix})$ **endif**

DEFINITION:
ok-c-after $(i,\ j,\ c,\ \mathit{matrix})$
$=$ **if** $i \simeq 0$ **then t**
elseif $(1+j) < \text{col}\,(\mathit{matrix})$
then tsquarep $(i-1,\ 1+j,\ \text{nth}\,(1+j,\ c),\ \mathit{matrix})$
$\wedge$ ($\neg$ tsquarep $(i-1,\ 1+j,\ 1+\text{nth}\,(1+j,\ c),\ \mathit{matrix})$)
$\wedge$ ok-c-after $(i,\ 1+j,\ c,\ \mathit{matrix})$
else t endif

DEFINITION:
zero-tail $(c,\ j,\ \mathit{col})$
$=$ **if** $j < \mathit{col}$ **then** $(\text{nth}\,(j,\ c) = 0) \wedge$ zero-tail $(c,\ 1+j,\ \mathit{col})$
else t endif

DEFINITION:
ok-c $(i,\ j,\ c,\ \mathit{matrix})$
$=$ (ok-c-before $(i,\ j,\ c,\ \mathit{matrix})$
$\wedge$ ok-c-after $(i,\ j,\ c,\ \mathit{matrix})$
$\wedge$ **if** $i \simeq 0$ **then** zero-tail $(c,\ j,\ \text{col}\,(\mathit{matrix}))$
else tsquarep $(i-1,\ j,\ \text{nth}\,(j,\ c),\ \mathit{matrix})$
$\wedge$ ($\neg$ tsquarep $(i-1,\ j,\ 1+\text{nth}\,(j,\ c),\ \mathit{matrix})$) **endif**)

THEOREM: put-preserves-ok-before
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (k \in \mathbf{N}) \wedge (j \not< k))$
$\rightarrow$ (ok-c-before $(i,\ k,\ \text{put}\,(j,\ \mathit{side},\ c),\ \mathit{matrix})$ = ok-c-before $(i,\ k,\ c,\ \mathit{matrix})$)

THEOREM: put-preserves-length
$(n < \text{length}\,(l)) \rightarrow (\text{length}\,(\text{put}\,(n,\ \mathit{val},\ l)) = \text{length}\,(l))$

THEOREM: put-preserves-ok-after
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (k \in \mathbf{N}) \wedge (j < k) \wedge (j < \text{col}\,(\mathit{matrix})))$
$\rightarrow$ (ok-c-after $(i,\ k,\ \text{put}\,(j,\ \mathit{side},\ c),\ \mathit{matrix})$ = ok-c-after $(i,\ k,\ c,\ \mathit{matrix})$)

DEFINITION:
next $(i,\ j,\ \mathit{col})$
$=$ **if** $(1+j) < \mathit{col}$ **then** cons $(i,\ 1+j)$
else cons $(1+i,\ 0)$ **endif**

```
;;; We are headed next toward proving that the accumulating
;;; vector C has the OK-C property preserved by PUTting in
;;; a new entry which is appropriate, i.e. which is the size
;;; of the largest square submatrix with the current coordinate
;;; at the lower right corner.  First we prove PUT-PRESERVES-OK,
;;; which shows that this is the case when SIDE is that
;;; appropriate size.  Then we define the function TSQ-LOCAL
;;; which actually computes this SIDE, and show that it really
;;; does so.
```

THEOREM: zero-tail-preserved

$(j < k) \rightarrow (\text{zero-tail}(\text{put}(j,\ side,\ c),\ k,\ col) = \text{zero-tail}(c,\ k,\ col))$

THEOREM: put-preserves-ok-1

$$\begin{aligned}
&((i \in \mathbf{N}) \\
&\ \wedge\ (j \in \mathbf{N}) \\
&\ \wedge\ \text{ok-c}(i,\ j,\ c,\ matrix) \\
&\ \wedge\ \text{tsquarep}(i,\ j,\ side,\ matrix) \\
&\ \wedge\ (\neg\ \text{tsquarep}(i,\ j,\ 1 + side,\ matrix)) \\
&\ \wedge\ ((1 + j) < \text{col}(matrix))) \\
&\rightarrow\ \text{ok-c}(\text{car}(\text{next}(i,\ j,\ \text{col}(matrix))), \\
&\qquad\quad \text{cdr}(\text{next}(i,\ j,\ \text{col}(matrix))), \\
&\qquad\quad \text{put}(j,\ side,\ c), \\
&\qquad\quad matrix)
\end{aligned}$$

THEOREM: ok-c-before-property

$$\begin{aligned}
&(\text{ok-c-before}(i,\ j,\ c,\ matrix) \wedge (k < j) \wedge (k \in \mathbf{N})) \\
&\rightarrow\ (\text{tsquarep}(i,\ k,\ \text{nth}(k,\ c),\ matrix) \\
&\qquad \wedge\ (\neg\ \text{tsquarep}(i,\ k,\ 1 + \text{nth}(k,\ c),\ matrix)))
\end{aligned}$$

THEOREM: put-preserves-ok-2

$$\begin{aligned}
&((i \in \mathbf{N}) \\
&\ \wedge\ (j \in \mathbf{N}) \\
&\ \wedge\ (k \in \mathbf{N}) \\
&\ \wedge\ \text{ok-c-before}(i - 1,\ j,\ c,\ matrix) \\
&\ \wedge\ \text{tsquarep}(i - 1,\ j,\ side,\ matrix) \\
&\ \wedge\ (\neg\ \text{tsquarep}(i - 1,\ j,\ 1 + side,\ matrix)) \\
&\ \wedge\ (j = (\text{col}(matrix) - 1)) \\
&\ \wedge\ (\text{col}(matrix) \neq 0)) \\
&\rightarrow\ \text{ok-c-after}(i,\ k,\ \text{put}(j,\ side,\ c),\ matrix)
\end{aligned}$$

THEOREM: put-preserves-ok

$$\begin{aligned}
&((i \in \mathbf{N}) \\
&\ \wedge\ (j \in \mathbf{N})
\end{aligned}$$

$$
\begin{array}{ll}
\wedge & \text{ok-c}\,(i,\ j,\ c,\ \mathit{matrix}) \\
\wedge & \text{tsquarep}\,(i,\ j,\ \mathit{side},\ \mathit{matrix}) \\
\wedge & (\neg\ \text{tsquarep}\,(i,\ j,\ 1 + \mathit{side},\ \mathit{matrix})) \\
\wedge & (j < \text{col}\,(\mathit{matrix}))) \\
\rightarrow & \text{ok-c}\,(\text{car}\,(\text{next}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
 & \quad \text{cdr}\,(\text{next}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
 & \quad \text{put}\,(j,\ \mathit{side},\ c), \\
 & \quad \mathit{matrix})
\end{array}
$$

Definition:

tsq-local $(i,\ j,\ c,\ \mathit{matrix})$

$$
\begin{array}{l}
= \ \textbf{if}\ \text{access}\,(\mathit{matrix},\ i,\ j) \\
\quad \textbf{then if}\ j \simeq 0\ \ \textbf{then}\ 1 \\
\qquad \textbf{elseif}\ \text{access}\,(\mathit{matrix}, \\
\qquad\qquad i - \min\,(\text{nth}\,(j-1,\ c),\ \text{nth}\,(j,\ c)), \\
\qquad\qquad j - \min\,(\text{nth}\,(j-1,\ c),\ \text{nth}\,(j,\ c))) \\
\qquad \textbf{then}\ 1 + \min\,(\text{nth}\,(j-1,\ c),\ \text{nth}\,(j,\ c)) \\
\qquad \textbf{else}\ \min\,(\text{nth}\,(j-1,\ c),\ \text{nth}\,(j,\ c))\ \textbf{endif} \\
\quad \textbf{else}\ 0\ \textbf{endif}
\end{array}
$$

Definition:

update-c $(i,\ j,\ c,\ \mathit{matrix}) = \text{put}\,(j,\ \text{tsq-local}\,(i,\ j,\ c,\ \mathit{matrix}),\ c)$

```
; In order to prove the correctness of TSQ-LOCAL, we need to show
; that joining together appropriately nearby square submatrices
; results in a submatrix of appropriate size.  The lemmas
; TSQUAREP-JOIN-WITHOUT-CORNER-1 and -2 show that one gets a
; TSQAREP of the common size (cf. the statement of problem),
; while TSQUAREP-JOIN-WITH-CORNER gives a square of side one greater
; when the opposite corner has value TRUE.  Recall that the point
; now is to relieve the following two hypotheses of
; PUT-PRESERVES-OK, (TSQUAREP I J SIDE MATRIX) and
; (NOT (TSQUAREP I J (ADD1 SIDE) MATRIX)), where here SIDE is
; computed by UPDATE-C to be (TSQ-LOCAL I J C MATRIX).
```

Theorem: tsquarep-join-without-corner-1

$$
\begin{array}{ll}
\multicolumn{2}{l}{(\text{tsquarep}\,(0,\ j-1,\ \mathit{side},\ \mathit{matrix}) \wedge \text{access}\,(\mathit{matrix},\ 0,\ j))} \\
\rightarrow & \text{tsquarep}\,(0,\ j,\ \mathit{side},\ \mathit{matrix})
\end{array}
$$

Theorem: tsquarep-join-without-corner-2

$$
\begin{array}{ll}
\multicolumn{2}{l}{((i \in \mathbf{N})} \\
\wedge & (i \neq 0) \\
\wedge & \text{tsquarep}\,(i-1,\ j,\ \mathit{side},\ \mathit{matrix}) \\
\wedge & \text{tsquarep}\,(i,\ j-1,\ \mathit{side},\ \mathit{matrix})
\end{array}
$$

$\wedge$ access $(matrix,\ i,\ j))$
$\rightarrow$ tsquarep $(i,\ j,\ side,\ matrix)$

THEOREM: segmentp-one-longer-base
segmentp $(i,\ j,\ \mathtt{1},\ matrix) =$ access $(matrix,\ i,\ j)$

THEOREM: segmentp-goes-down-special-case
(segmentp $(i,\ j-1,\ left,\ matrix) \wedge$ access $(matrix,\ i,\ j))$
$\rightarrow$ segmentp $(i,\ j,\ left,\ matrix)$

THEOREM: segmentp-one-longer
$((i \in \mathbf{N})$
$\wedge$ $(j \in \mathbf{N})$
$\wedge$ segmentp $(i,\ j,\ left,\ matrix)$
$\wedge$ access $(matrix,\ i,\ j - left))$
$\rightarrow$ segmentp $(i,\ j,\ 1 + left,\ matrix)$

THEOREM: rectanglep-join-with-corner
(rectanglep $(i-1,\ j,\ left,\ up,\ matrix)$
$\wedge$ rectanglep $(i,\ j-1,\ left,\ up,\ matrix)$
$\wedge$ access $(matrix,\ i - up,\ j - left)$
$\wedge$ $(j \not< left)$
$\wedge$ $(i \not< up)$
$\wedge$ $(left \in \mathbf{N})$
$\wedge$ $(left \neq \mathtt{0})$
$\wedge$ $(up \in \mathbf{N})$
$\wedge$ $(up \neq \mathtt{0})$
$\wedge$ access $(matrix,\ i,\ j))$
$\rightarrow$ rectanglep $(i,\ j,\ 1 + left,\ 1 + up,\ matrix)$

THEOREM: tsquarep-join-with-corner
(tsquarep $(i-1,\ j,\ side,\ matrix)$
$\wedge$ tsquarep $(i,\ j-1,\ side,\ matrix)$
$\wedge$ access $(matrix,\ i - side,\ j - side)$
$\wedge$ access $(matrix,\ i,\ j)$
$\wedge$ $(i \in \mathbf{N})$
$\wedge$ $(i \neq \mathtt{0})$
$\wedge$ $(j \in \mathbf{N})$
$\wedge$ $(j \neq \mathtt{0}))$
$\rightarrow$ tsquarep $(i,\ j,\ 1 + side,\ matrix)$

```
; So now finally we show that UPDATE-C yields the appropriate SIDE
; for the hypotheses of PUT-PRESERVES-OK by applying the joining
; lemmas above.  First, a useful lemma.
```

THEOREM: ok-c-0
$(\text{ok-c}(\mathtt{0}, j, c, matrix) \land (j < \text{col}(matrix)))$
$\rightarrow \quad (\text{tsq-local}(\mathtt{0}, j, c, matrix)$
$\quad\quad = \ \textbf{if } \text{access}(matrix, \mathtt{0}, j) \textbf{ then } \mathtt{1}$
$\quad\quad\quad \textbf{else } \mathtt{0} \textbf{ endif})$

THEOREM: tsquarep-preserved-by-tsq-local-for-i-zero
$((j \in \mathbf{N}) \land \text{ok-c}(\mathtt{0}, j, c, matrix) \land (j < \text{col}(matrix)))$
$\rightarrow \quad \text{tsquarep}(\mathtt{0}, j, \text{tsq-local}(\mathtt{0}, j, c, matrix), matrix)$

THEOREM: tsq-local-tsquarep-bound-0
$((j \in \mathbf{N}) \land \text{ok-c}(\mathtt{0}, j, c, matrix) \land (j < \text{col}(matrix)))$
$\rightarrow \quad (\neg \text{tsquarep}(\mathtt{0}, j, 1 + \text{tsq-local}(\mathtt{0}, j, c, matrix), matrix))$

```
; For the case where I is non-zero in showing that TSQ-LOCAL has the
; desired property, the following lemma is useful.
```

THEOREM: ok-c-main-property
$((j \in \mathbf{N}) \land (j \neq \mathtt{0}) \land (i \in \mathbf{N}) \land (i \neq \mathtt{0}) \land \text{ok-c}(i, j, c, matrix))$
$\rightarrow \quad (\text{tsquarep}(i - 1, j, \min(\text{nth}(j - 1, c), \text{nth}(j, c)), matrix)$
$\quad\quad \land \quad \text{tsquarep}(i, j - 1, \min(\text{nth}(j - 1, c), \text{nth}(j, c)), matrix))$

THEOREM: tsquarep-preserved-by-tsq-local-for-i-non-zero
$((j \in \mathbf{N})$
$\land \quad (i \in \mathbf{N})$
$\land \quad (i \neq \mathtt{0})$
$\land \quad \text{ok-c}(i, j, c, matrix)$
$\land \quad (j < \text{col}(matrix)))$
$\rightarrow \quad \text{tsquarep}(i, j, \text{tsq-local}(i, j, c, matrix), matrix)$

```
; It remains then to show that (add1 (tsq-local ...)) is too large a side
; in the non-zero case.
```

THEOREM: access-i-non-numberp
$(i \notin \mathbf{N}) \rightarrow (\text{access}(matrix, i, j) = \text{access}(matrix, \mathtt{0}, j))$

THEOREM: access-j-non-numberp
$(j \notin \mathbf{N}) \rightarrow (\text{access}(matrix, i, j) = \text{access}(matrix, i, \mathtt{0}))$

THEOREM: segmentp-gives-every-access
$(\text{segmentp}(i, j, left, matrix) \land (left1 < left))$
$\rightarrow \quad \text{access}(matrix, i, j - left1)$

THEOREM: rectanglep-gives-every-access
(rectanglep $(i,\ j,\ \mathit{left},\ \mathit{up},\ \mathit{matrix}) \land (\mathit{up1} < \mathit{up}) \land (\mathit{left1} < \mathit{left}))$
$\rightarrow$ access $(\mathit{matrix},\ i - \mathit{up1},\ j - \mathit{left1})$

```
; and a clear corollary:
```

THEOREM: tsquarep-gives-every-access
(tsquarep $(i,\ j,\ \mathit{side},\ \mathit{matrix}) \land (\mathit{side1} < \mathit{side}))$
$\rightarrow$ access $(\mathit{matrix},\ i - \mathit{side1},\ j - \mathit{side1})$

THEOREM: ok-c-main-bound-property
$((j \in \mathbf{N})$
$\land$ $(j \neq 0)$
$\land$ $(i \in \mathbf{N})$
$\land$ $(i \neq 0)$
$\land$ ok-c $(i,\ j,\ c,\ \mathit{matrix})$
$\land$ tsquarep $(i - 1,\ j,\ 1 + \min(\mathrm{nth}(j - 1,\ c),\ \mathrm{nth}(j,\ c)),\ \mathit{matrix}))$
$\rightarrow$ $(\neg$ tsquarep $(i,\ j - 1,\ 1 + \min(\mathrm{nth}(j - 1,\ c),\ \mathrm{nth}(j,\ c)),\ \mathit{matrix}))$

THEOREM: rectanglep-goes-down-in-i-and-up
$((i \in \mathbf{N}) \land (j \in \mathbf{N}) \land$ rectanglep $(i,\ j,\ \mathit{left},\ 1 + \mathit{up},\ \mathit{matrix}))$
$\rightarrow$ rectanglep $(i - 1,\ j,\ \mathit{left},\ \mathit{up},\ \mathit{matrix})$

THEOREM: rectanglep-goes-down-in-j-and-left
$((i \in \mathbf{N}) \land (j \in \mathbf{N}) \land$ rectanglep $(i,\ j,\ 1 + \mathit{left},\ \mathit{up},\ \mathit{matrix}))$
$\rightarrow$ rectanglep $(i,\ j - 1,\ \mathit{left},\ \mathit{up},\ \mathit{matrix})$

THEOREM: tsquarep-goes-down-in-coordinates-and-side
$((i \in \mathbf{N})$
$\land$ $(i \neq 0)$
$\land$ $(j \in \mathbf{N})$
$\land$ $(j \neq 0)$
$\land$ $(\mathit{side} \in \mathbf{N})$
$\land$ tsquarep $(i,\ j,\ 1 + \mathit{side},\ \mathit{matrix}))$
$\rightarrow$ (tsquarep $(i,\ j - 1,\ \mathit{side},\ \mathit{matrix}) \land$ tsquarep $(i - 1,\ j,\ \mathit{side},\ \mathit{matrix}))$

THEOREM: tsq-local-tsquarep-bound-non-0
$((i \in \mathbf{N})$
$\land$ $(i \neq 0)$
$\land$ $(j \in \mathbf{N})$
$\land$ ok-c $(i,\ j,\ c,\ \mathit{matrix})$
$\land$ $(j < \mathrm{col}(\mathit{matrix})))$
$\rightarrow$ $(\neg$ tsquarep $(i,\ j,\ 1 +$ tsq-local $(i,\ j,\ c,\ \mathit{matrix}),\ \mathit{matrix}))$

```
; Here then is what we've been aiming at:
```

THEOREM: update-c-preserves-ok-c
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge \text{ok-c}(i, j, c, \mathit{matrix}) \wedge (j < \text{col}(\mathit{matrix})))$
$\rightarrow \quad \text{ok-c}(\text{car}(\text{next}(i, j, \text{col}(\mathit{matrix}))),$
$\qquad \text{cdr}(\text{next}(i, j, \text{col}(\mathit{matrix}))),$
$\qquad \text{update-c}(i, j, c, \mathit{matrix}),$
$\qquad \mathit{matrix})$

DEFINITION:
update-acc $(i, j, \mathit{new}, \mathit{acc})$
$=$ **if** car $(\mathit{acc}) < \mathit{new}$ **then** list (new, i, j)
else acc **endif**

DEFINITION:
lex2 $(\mathit{l1}, \mathit{l2})$
$= \quad ((\text{car}(\mathit{l1}) < \text{car}(\mathit{l2}))$
$\quad \vee \quad ((\text{car}(\mathit{l1}) = \text{car}(\mathit{l2})) \wedge (\text{cadr}(\mathit{l1}) < \text{cadr}(\mathit{l2}))))$

DEFINITION:
tsq-rec $(i, j, c, \mathit{matrix}, \mathit{acc})$
$=$ **if** lex2 (list (i, j), list (length (matrix), 0))
then tsq-rec (car (next $(i, j,$ col $(\mathit{matrix})))$,
cdr (next $(i, j,$ col $(\mathit{matrix})))$,
update-c $(i, j, c, \mathit{matrix})$,
matrix,
update-acc $(i, j,$ tsq-local $(i, j, c, \mathit{matrix}), \mathit{acc}))$
else acc **endif**

DEFINITION: c-init (matrix) = zeroes (col $(\mathit{matrix}) - 1)$

DEFINITION:
previous (i, j, col)
$=$ **if** $j \simeq 0$
then if $i \simeq 0$ **then** cons (0, 0)
else cons $(i - 1, \mathit{col} - 1)$ **endif**
else cons $(i, j - 1)$ **endif**

THEOREM: previous-next
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (j < \mathit{col}))$
$\rightarrow \quad (\text{previous}(\text{car}(\text{next}(i, j, \mathit{col})), \text{cdr}(\text{next}(i, j, \mathit{col})), \mathit{col}) = \text{cons}(i, j))$

DEFINITION:
bound $(i, j, \mathit{matrix}, \mathit{size})$

$$
\begin{aligned}
= \quad & \textbf{if } (i \simeq 0) \wedge (j \simeq 0) \textbf{ then t} \\
& \textbf{else } (\neg\ \text{tsquarep}\,(\text{car}\,(\text{previous}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad\qquad \text{cdr}\,(\text{previous}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad\qquad 1 + \mathit{size}, \\
& \qquad\qquad \mathit{matrix})) \\
& \quad \wedge \quad \text{bound}\,(\text{car}\,(\text{previous}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad\qquad \text{cdr}\,(\text{previous}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad\qquad \mathit{matrix}, \\
& \qquad\qquad \mathit{size}) \textbf{ endif}
\end{aligned}
$$

DEFINITION:
in-matrix $(i,\ j,\ \mathit{matrix})$

$$
\begin{aligned}
= \quad & ((i \in \mathbf{N}) \\
& \wedge \quad (j \in \mathbf{N}) \\
& \wedge \quad (i < \text{length}\,(\mathit{matrix})) \\
& \wedge \quad (j < \text{col}\,(\mathit{matrix})))
\end{aligned}
$$

```
; The main lemma remaining on the way to
; INVARIANT-IS-PRESERVED-BY-CALL-OF-TSQ-REC is the following:
```

THEOREM: bound-goes-up
$(\text{bound}\,(i,\ j,\ \mathit{matrix},\ b) \wedge (\mathit{b1} \not< b)) \rightarrow \text{bound}\,(i,\ j,\ \mathit{matrix},\ \mathit{b1})$

THEOREM: lessp-update-acc

$$
\begin{aligned}
& ((\text{car}\,(\text{update-acc}\,(i,\ j,\ \mathit{anything},\ \mathit{acc})) < \mathit{anything}) = \mathbf{f}) \\
\wedge \quad & ((\text{car}\,(\text{update-acc}\,(i,\ j,\ \mathit{anything},\ \mathit{acc})) < \text{car}\,(\mathit{acc})) = \mathbf{f})
\end{aligned}
$$

THEOREM: bound-preserved-by-update-acc

$$
\begin{aligned}
& ((i \in \mathbf{N}) \\
\wedge \quad & (j \in \mathbf{N}) \\
\wedge \quad & \text{ok-c}\,(i,\ j,\ c,\ \mathit{matrix}) \\
\wedge \quad & (j < \text{col}\,(\mathit{matrix})) \\
\wedge \quad & \text{bound}\,(i,\ j,\ \mathit{matrix},\ \text{car}\,(\mathit{acc}))) \\
\rightarrow \quad & \text{bound}\,(\text{car}\,(\text{next}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad \text{cdr}\,(\text{next}\,(i,\ j,\ \text{col}\,(\mathit{matrix}))), \\
& \qquad \mathit{matrix}, \\
& \qquad \text{car}\,(\text{update-acc}\,(i,\ j,\ \text{tsq-local}\,(i,\ j,\ c,\ \mathit{matrix}),\ \mathit{acc})))
\end{aligned}
$$

THEOREM: car-of-next-numberp
$(i \in \mathbf{N}) \rightarrow (\text{car}\,(\text{next}\,(i,\ j,\ \mathit{col})) \in \mathbf{N})$

THEOREM: cdr-of-next-numberp
$\text{cdr}\,(\text{next}\,(i,\ j,\ \mathit{col})) \in \mathbf{N}$

THEOREM: cdr-of-next-lessp
$(x < col) \rightarrow ((\mathrm{cdr}(\mathrm{next}(i, j, col)) < col) = \mathbf{t})$

THEOREM: lex2-next
$\mathrm{lex2}(\mathrm{list}(i, j), \mathrm{list}(\mathrm{car}(\mathrm{next}(i, j, col)), \mathrm{cdr}(\mathrm{next}(i, j, col))))$

THEOREM: lex2-transitivity
$(\mathrm{lex2}(x, y) \wedge \mathrm{lex2}(y, z)) \rightarrow \mathrm{lex2}(x, z)$

DEFINITION:
all-f $(i, j, matrix)$
$=$ **if** $(i \simeq 0) \wedge (j \simeq 0)$ **then** $\mathbf{t}$
else $(\neg\, \mathrm{access}(matrix,$
$\mathrm{car}(\mathrm{previous}(i, j, \mathrm{col}(matrix))),$
$\mathrm{cdr}(\mathrm{previous}(i, j, \mathrm{col}(matrix)))))$
$\wedge$ all-f $(\mathrm{car}(\mathrm{previous}(i, j, \mathrm{col}(matrix))),$
$\mathrm{cdr}(\mathrm{previous}(i, j, \mathrm{col}(matrix))),$
$matrix)$ **endif**

DEFINITION:
invariant $(i, j, c, matrix, acc)$
$= ((i \in \mathbf{N})$
$\wedge$ $(j \in \mathbf{N})$
$\wedge$ in-matrix $(\mathrm{cadr}(acc), \mathrm{caddr}(acc), matrix)$
$\wedge$ $(j < \mathrm{col}(matrix))$
$\wedge$ ok-c $(i, j, c, matrix)$
$\wedge$ $((\text{all-f}(i, j, matrix) \wedge (acc = \mathrm{list}(0, 0, 0)))$
$\vee$ tsquarep $(\mathrm{cadr}(acc), \mathrm{caddr}(acc), \mathrm{car}(acc), matrix))$
$\wedge$ $(\mathrm{lex2}(\mathrm{cdr}(acc), \mathrm{list}(i, j))$
$\vee$ $((i = 0) \wedge (j = 0) \wedge (acc = \mathrm{list}(0, 0, 0))))$
$\wedge$ bound $(i, j, matrix, \mathrm{car}(acc)))$

THEOREM: invariant-is-preserved-by-call-of-tsq-rec
$(\mathrm{lex2}(\mathrm{list}(i, j), \mathrm{cons}(\mathrm{length}(matrix), \texttt{'(0)})) \wedge \mathrm{invariant}(i, j, c, matrix, acc))$
$\rightarrow$ invariant $(\mathrm{car}(\mathrm{next}(i, j, \mathrm{col}(matrix))),$
$\mathrm{cdr}(\mathrm{next}(i, j, \mathrm{col}(matrix))),$
update-c $(i, j, c, matrix),$
$matrix,$
update-acc $(i, j, \text{tsq-local}(i, j, c, matrix), acc))$

THEOREM: tsq-rec-correctness-from-invariant-base
invariant $(i, j, c, matrix, acc)$
$\rightarrow$ (tsquarep $(\mathrm{cadr}(acc), \mathrm{caddr}(acc), \mathrm{car}(acc), matrix)$
$\wedge$ in-matrix $(\mathrm{cadr}(acc), \mathrm{caddr}(acc), matrix)$
$\wedge$ bound $(i, j, matrix, \mathrm{car}(acc)))$

THEOREM: bound-lex2

$$(\text{bound}(i, j, matrix, b) \land (i \in \mathbf{N}) \land (j \in \mathbf{N}) \land (i1 \in \mathbf{N}) \land (j1 \in \mathbf{N}) \land (j1 < \text{col}(matrix)) \land (\neg\, \text{lex2}(\text{list}(i, j), \text{list}(i1, j1)))) \rightarrow \text{bound}(i1, j1, matrix, b)$$

THEOREM: tsq-rec-correctness-from-invariant

$$\text{invariant}(i, j, c, matrix, acc) \rightarrow (\text{tsquarep}(\text{cadr}(\text{tsq-rec}(i, j, c, matrix, acc)), \text{caddr}(\text{tsq-rec}(i, j, c, matrix, acc)), \text{car}(\text{tsq-rec}(i, j, c, matrix, acc)), matrix) \land \text{in-matrix}(\text{cadr}(\text{tsq-rec}(i, j, c, matrix, acc)), \text{caddr}(\text{tsq-rec}(i, j, c, matrix, acc)), matrix) \land \text{bound}(\text{length}(matrix), 0, matrix, \text{car}(\text{tsq-rec}(i, j, c, matrix, acc))))$$

DEFINITION:

$$\text{tsq-i-coord}(matrix) = \text{cadr}(\text{tsq-rec}(0, 0, \text{c-init}(matrix), matrix, \text{list}(0, 0, 0)))$$

DEFINITION:

$$\text{tsq-j-coord}(matrix) = \text{caddr}(\text{tsq-rec}(0, 0, \text{c-init}(matrix), matrix, \text{list}(0, 0, 0)))$$

DEFINITION:

$$\text{tsq}(matrix) = \text{car}(\text{tsq-rec}(0, 0, \text{c-init}(matrix), matrix, \text{list}(0, 0, 0)))$$

THEOREM: zero-tail-zeroes-lemma

$$\text{nth}(i, \text{zeroes}(n)) = 0$$

THEOREM: zero-tail-zeroes

$$\text{zero-tail}(\text{zeroes}(n), i, j)$$

THEOREM: correctness-of-tsq

$$((\text{length}(matrix) \neq 0) \land (\text{col}(matrix) \neq 0)) \rightarrow (\text{tsquarep}(\text{tsq-i-coord}(matrix), \text{tsq-j-coord}(matrix), \text{tsq}(matrix), matrix) \land \text{in-matrix}(\text{tsq-i-coord}(matrix), \text{tsq-j-coord}(matrix), matrix) \land \text{bound}(\text{length}(matrix), 0, matrix, \text{tsq}(matrix)))$$

Index