EVENT: Start with the initial **nqthm** theory.

;;; The proof in here is based on the hint to exercise (8) on page 43 of Kunen’s Set Theory book.

;;; Imagine sets a and b and corresponding functions fa and fb which map a to b and vice-versa, where both are one-one.

**Definition:** \( \text{id}(x) = x \)

**Conservative Axiom: fa-and-fb-are-one-one**

\[
\begin{align*}
(a(x) \land a(y) \land (x \neq y)) &\rightarrow (fa(x) \neq fa(y)) \\
\land \ (b(x) \land b(y) \land (x \neq y)) &\rightarrow (fb(x) \neq fb(y)) \\
\land \ (a(x) \rightarrow b(fa(x))) \\
\land \ (b(x) \rightarrow a(fb(x))) \\
\land \ (\text{truep}(a(x)) \lor \text{falsep}(a(x))) \\
\land \ (\text{truep}(b(x)) \lor \text{falsep}(b(x)))
\end{align*}
\]

Simultaneously, we introduce the new function symbols \(fa, fb, a,\) and \(b\).

**Definition:**

\[
\text{in-fa-range}(x) \leftrightarrow \exists \ fa-1 \ (a(fa-1) \land (fa(fa-1) = x))
\]

;;; The following 3 events were generated automatically to help with the proof-checker’s application of the macro command SK*.

EVENT: Disable in-fa-range.

**Theorem:** in-fa-range-suff

\[
(a(fa-1) \land (fa(fa-1) = x)) \rightarrow \text{in-fa-range}(x)
\]

**Theorem:** in-fa-range-necc

\[
(\neg (a(fa-1(x)) \land (fa(fa-1(x)) = x))) \rightarrow (\neg \text{in-fa-range}(x))
\]

**Definition:**

\[
\text{in-fb-range}(x) \leftrightarrow \exists \ fb-1 \ (b(fb-1) \land (fb(fb-1) = x))
\]

;;; The following 3 events were generated automatically to help with the proof-checker’s application of the macro command sk*.

EVENT: Disable in-fb-range.
Theorem: in-fb-range-suff
\((b(\text{fb-1}) \land (\text{fb}(\text{fb-1}) = x)) \rightarrow \text{in-fb-range}(x)\)

Theorem: in-fb-range-necc
\((\neg (b(\text{fb-1}(x)) \land (\text{fb}(\text{fb-1}(x)) = x))) \rightarrow (\neg \text{in-fb-range}(x))\)

Definition:
circled (flg, x, n) =
  if flg = 'a then if n \simeq 0 then a(x)
  else in-fb-range (x) \land circled ('b, fb-1 (x), n - 1) endif
  elseif n \simeq 0 then b(x)
  else in-fa-range (x) \land circled ('a, fa-1 (x), n - 1) endif

Definition:
a-core (x) \leftrightarrow (a(x) \land \forall a\text{-level} (((a\text{-level} \in \mathbb{N}) \land circled ('a, x, a\text{-level})) \rightarrow circled ('a, x, 1 + a\text{-level}))

Event: Disable a-core.

Theorem: a-core-necc-base
\(((n \simeq 0) \land a-core(x)) \rightarrow circled ('a, x, n)\)

Theorem: a-core-necc-induction
\(((n \not\simeq 0) \land a-core(x) \land circled ('a, x, n - 1)) \rightarrow circled ('a, x, n)\)

Theorem: a-core-necc
\((\neg circled ('a, x, n)) \rightarrow (\neg a-core(x))\)

Event: Disable a-core-necc-base.

Event: Disable a-core-necc-induction.

Theorem: a-core-suff
\((a(x) \land ((a\text{-level}(x) \in \mathbb{N}) \land circled ('a, x, a\text{-level}(x)))) \rightarrow circled ('a, x, 1 + a\text{-level}(x)))\)
\rightarrow a-core (x)

Definition:
b-core (x) \leftrightarrow (b(x) \land \forall b\text{-level} (((b\text{-level} \in \mathbb{N}) \land circled ('b, x, b\text{-level})) \rightarrow circled ('b, x, 1 + b\text{-level})))
EVENT: Disable b-core.

**Theorem:** b-core-necc-base

\(((n \simeq 0) \land \text{b-core}(x)) \rightarrow \text{circled}('b, x, n)\)

**Theorem:** b-core-necc-induction

\(((n \not\simeq 0) \land \text{b-core}(x) \land \text{circled}('b, x, n - 1)) \rightarrow \text{circled}('b, x, n)\)

**Theorem:** b-core-necc

\((\neg \text{circled}('b, x, n)) \rightarrow (\neg \text{b-core}(x))\)

EVENT: Disable b-core-necc-base.

EVENT: Disable b-core-necc-induction.

**Theorem:** b-core-suff

\((\text{b}(x) \land ((\text{b-level}(x) \in \mathbb{N}) \land \text{circled}('b, x, \text{b-level}(x)))) \rightarrow \text{circled}('b, x, 1 + \text{b-level}(x)))) \rightarrow \text{b-core}(x)\)

**Definition:**

\(\text{parity}(n) = \text{if } n \simeq 0 \text{ then } t \text{ else } \neg \text{parity}(n - 1) \text{ endif}\)

**Definition:**

\(j(x) = \text{if } \text{a-core}(x) \lor \text{parity}(\text{a-level}(x)) \text{ then } \text{fa}(x) \text{ else } \text{fb-1}(x) \text{ endif}\)

**Definition:**

\(j-1(y) = \text{if } \text{b-core}(y) \lor (\neg \text{parity}(\text{b-level}(y))) \text{ then } \text{fa-1}(y) \text{ else } \text{fb}(y) \text{ endif}\)

; Our goals:

; (prove-lemma j-1-j (rewrite)
;  (implies (a x)
;    (equal (j-1 (j x)) x)))

; (prove-lemma j-j-1 (rewrite)
;  (implies (b y)
;    (equal (j (j-1 y)) y)))
We'll start on the first of these. The theorem-prover output suggests the following lemma:

(prove-lemma b-core-fa (rewrite)
  (implies (a x)
    (iff (b-core (fa x))
      (a-core x))))

A main lemma is that (a-core x) => (b-core (fa x)) for x in a.

**Theorem**: fa-1-inverts-fa

\[ a(x) \rightarrow (fa-1 (fa(x)) = x) \]

The following is useful for proof-checker rewriting:

**Theorem**: in-fa-range-fa

\[ a(x) \rightarrow in-fa-range (fa(x)) \]

**Theorem**: b-core-fa

\[ a(x) \rightarrow (b-core (fa(x)) \leftrightarrow a-core(x)) \]

On to Case 2 of j-1-j. We find there the following contradictory hyps:

(and (a x)
  (parity (a-level x))
  (not (a-core x))
  (parity (b-level (fa x))))

We want to prove:

**Theorem**: b-level-fa

\( (\text{implies} \ (\text{and} \ (a \ x) \ (\not b \ y)) \rightarrow (\text{fa} \ x) \not= y) \)

**Theorem**: fa-range-contained-in-b

\( (\not b \ y) \rightarrow (\not in-fa-range \ y) \)
Theorem: circled-implies-b
\[ \text{circled}(\bar{b}, y, n) \rightarrow b(y) \]

Theorem: a-fb-equality-rewrite
\[ (b(y) \land (\neg a(x))) \rightarrow (fb(y) \neq x) \]

Theorem: fb-range-contained-in-a
\[ (\neg a(x)) \rightarrow (\neg \text{in-fb-range}(x)) \]

Theorem: circled-implies-a
\[ \text{circled}(\bar{a}, y, n) \rightarrow a(y) \]

Theorem: circled-monotone
\[ (\text{circled}(fg, x, j) \land (j \neq i)) \rightarrow \text{circled}(fg, x, i) \]

Theorem: circled-b-fa
\[ a(x) \rightarrow (\text{circled}(\bar{b}, \text{fa}(x), n) = \text{circled}(\bar{a}, x, n - 1)) \]

Theorem: b-level-fa-hack-lemma
\[ (\text{b-level}(\text{fa}(x)) \in \mathbb{N}) \rightarrow (\text{b-level}(\text{fa}(x)) = (1 + \text{a-level}(x))) \]

\[ = (\neg ((\text{b-level}(\text{fa}(x)) < (1 + \text{a-level}(x)))) \lor ((1 + \text{a-level})(x) < \text{b-level}(\text{fa}(x)))))) \]

Theorem: b-level-fa
\[ (a(x) \land (\neg \text{a-core}(x))) \rightarrow \text{b-level}(\text{fa}(x)) = (1 + \text{a-level}(x))) \]

;; On to Case 3 of j-1-j. Our first subgoal is:

;; \text{IMPLIES} (\text{AND} (A X))
;; \text{;} (\neg (\text{A-CORE} X))
;; \text{;} (\neg (\text{PARITY} (A-LEVEL X)))
;; \text{;} (\neg (B-CORE (FB-1 X)))
;; \text{;} (\text{PARITY} (B-LEVEL (FB-1 X))))
;; \text{;} (\text{EQUAL} (FB (FB-1 X)) X))

;; and this follows from the following two lemmas.

Theorem: not-parity-a-level-implies-in-fb-range
\[ (a(x) \land (\neg \text{parity}(\text{a-level}(x)))) \rightarrow \text{in-fb-range}(x) \]

;; The next subcase of j-1-j is:

;(\text{implies} (\text{and} (a x))
;(\text{not} (\text{a-core} x))

;; 5
(not (parity (a-level x)))

(implies (and (a x)
              (not (a-core x))
              (in-fb-range x))
          (equal (b-level (fb-1 x))
                 (sub1 (a-level x))))

;; I’ve proved nearly the analogous thing already for fa. Let’s prove that
;; one for fb and then deduce this from it.

THEOREM: fb-1-inverts-fb
b (x) → (fb-1 (fb (x)) = x)

THEOREM: in-fb-range-fb
b (x) → in-fb-range (fb (x))

THEOREM: circled-a-fb
b (x) → (circled (’a, fb (x), n) = circled (’b, x, n − 1))

THEOREM: a-level-fb-hack-lemma
(a-level (fb (x)) ∈ N)
→ ((a-level (fb (x)) = (1 + b-level (x)))
   = (¬ ((a-level (fb (x)) < (1 + b-level (x)))
        ∨ ((1 + b-level (x)) < a-level (fb (x)))))

THEOREM: a-core-fb
b (x) → (a-core (fb (x)) ↔ b-core (x))

THEOREM: a-level-fb
(b (x) ∧ (¬ b-core (x))) → (a-level (fb (x)) = (1 + b-level (x)))

THEOREM: b-level-fb-1
(a (x) ∧ (¬ a-core (x)) ∧ in-fb-range (x))
→ (b-level (fb-1 (x)) = (a-level (x) − 1))

;; It remains only to prove the following subcase, and then we’re done with j-1-j:

(implies (and (a x)
Theorem: b-core-implies-b
b-core (x) → b (x)

Theorem: fb-fb-1
(a (x) ∧ in-fb-range (x)) → (fb (fb-1 (x)) = x)

Theorem: b-core-fb-1
(a (x) ∧ (¬ a-core (x)) ∧ (¬ parity (a-level (x)))) → (¬ b-core (fb-1 (x)))

;; Now finally for our first main goal:

Theorem: j-1-j
a (x) → (j-1 (j (x)) = x)

;; We're ready now for the converse. Here's the first sticking point:

;; (implies (and (b-core y)
;; (a-core (fa-1 y)))
;; (equal (fa (fa-1 y)) y))

;; We need only the following two easy lemmas:

Theorem: b-core-implies-in-fa-range
b-core (y) → in-fa-range (y)

Theorem: fa-fa-1
in-fa-range (y) → (fa (fa-1 (y)) = y)

;; Our next goal is:

;; (implies (and (b-core y)
;; (not (a-core (fa-1 y)))
;; (not (parity (a-level (fa-1 y))))
;; (equal (fb-1 (fa-1 y)) y))

;; which is taken care of simply by:
Theorem: a-core-fa-1
\[
\text{in-fa-range} (y) \rightarrow (\text{a-core} (\text{fa-1} (y)) \leftrightarrow \text{b-core} (y))
\]

;; next we have to prove

;(implies (and (b y)
    ; (not (parity (b-level y)))
    ; (a-core (fa-1 y)))
    ; (equal (fa (fa-1 y)) y))

;; which follows from fa-fa-1 together with an obvious analog of
;; NOT-PARITY-A-LEVEL-IMPLIES-IN-FB-RANGE:

Theorem: not-parity-b-level-implies-in-fa-range
\[
(b (y) \land (\neg \text{parity} (\text{b-level} (y)))) \rightarrow \text{in-fa-range} (y)
\]

;; It remains only to prove:

;(implies (and (b y)
    ; (not (parity (b-level y)))
    ; (not (b-core y))
    ; (not (parity (a-level (fa-1 y)))))
    ; (equal (fb-1 (fa-1 y)) y))

;; which follows from an analogue of B-LEVEL-FB-1:

Theorem: a-level-fa-1
\[
(b (y) \land (\neg \text{b-core} (y)) \land \text{in-fa-range} (y))
\rightarrow (\text{a-level} (\text{fa-1} (y)) = (\text{b-level} (y) - 1))
\]

;;; and we’re done!!

Theorem: j-j-1
\[
\text{b} (y) \rightarrow (j (j-1 (y)) = y)
\]

;; To summarize:

;; From the axiom saying that fa maps a one-one into b and fb maps b one-one into a,

;(constrain fa-and-fb-are-one-one (rewrite)
    ; (and (implies (and (a x) (a y)) (not (equal x y)))
    ; (not (equal (fa x) (fa y)))))
(implies (and (b x) (b y) (not (equal x y)))
; (not (equal (fb x) (fb y))))
; (implies (a x) (b (fa x)))
; (implies (b x) (a (fb x))))
; (or (truep (a x)) (falsep (a x)))
; (or (truep (b x)) (falsep (b x))))
; ((fa id) (fb id) (a (lambda (x) t)) (b (lambda (x) t))))

;; to finish this off we simply prove the obvious lemmas needed for
;; j-iso below. For the first, j-range,
;; we need a lemma (as seen from observing the failed proof transcript).

**THEOREM:** in-fb-range-implies-b-fb-1
in-fb-range (y) → b (fb-1 (y))

**THEOREM:** j-range
a (x) → b (j (x))

**THEOREM:** in-fa-range-implies-a-fa-1
in-fa-range (y) → a (fa-1 (y))

**THEOREM:** j-1-range
b (x) → a (j-1 (x))

**EVENT:** Disable j.

**EVENT:** Disable j-1.

**THEOREM:** j-is-one-one
(a (x1) ∧ a (x2) ∧ (x1 ≠ x2)) → (j (x1) ≠ j (x2))

;; we were able to conservatively extend the theory culminating in
;; definitions of a function j which maps a one-one into b:

**DEFINITION:**
J-ISO
⇔ (∀ x (a (x) → b (j (x))))
    ∧ (∀ x1, x2 ((a (x1) ∧ a (x2) ∧ (j (x1)) = j (x2)))
        → (x1 = x2))
    ∧ (∀ y (b (y) → ∃ x (a (x) ∧ (j (x) = y))))

**THEOREM:** j-is-an-isomorphism
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