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;; The infinite Ramsey Theorem for exponent 2.
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EVENT: Start with the initial **nqthm** theory.

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;; We introduce the partition p on pairs via a CONSTRAIN event which
;; asserts that p is a symmetric function with range {1, ...,
;; (bound)}.  Actually, we do this by (roughly speaking) introducing a
;; function p-num on numbers and then extending it to a function p on
;; everything.
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CONSERVATIVE AXIOM: p-num-intro

$((x \in \mathbf{N}) \wedge (y \in \mathbf{N}))$
$\rightarrow \quad ((0 < \text{p-num}(x, y))$
$\quad \wedge \quad (\text{BOUND} \not< \text{p-num}(x, y))$
$\quad \wedge \quad (\text{p-num}(x, y) = \text{p-num}(y, x)))$

Simultaneously, we introduce the new function symbols *p-num* and *bound*.

EVENT: Disable p-num-intro.

DEFINITION: $\text{p}(x, y) = \text{p-num}(\text{fix}(x), \text{fix}(y))$

THEOREM: p-intro

$(0 < \text{p}(x, y)) \wedge (\text{BOUND} \not< \text{p}(x, y)) \wedge (\text{p}(x, y) = \text{p}(y, x))$

EVENT: Disable p.

DEFINITION:

prehom-seq-1 (a, x)
= **if** listp (x)
  then (p (caar (x), a) = cdar (x)) $\wedge$ prehom-seq-1 $(a$, cdr $(x))$
  else t endif

DEFINITION:

prehom-seq (x)
= **if** listp (x)
  then if listp (cdr (x))
    then (car (cadr (x)) < car (car (x)))
      $\wedge$ prehom-seq-1 (caar (x), cdr (x))
      $\wedge$ prehom-seq (cdr (x))
    else t endif
  else t endif

DEFINITION:
extensible $(s) \leftrightarrow \forall\, above\ \exists\, next\ ((above < next) \land \text{prehom-seq-1}\,(next,\ s))$

```
;; The following two lemmas are for the benefit of the proof-checker
;; macro-command SK*.
```

THEOREM: extensible-suff
$((\text{above}\,(s) < next) \land \text{prehom-seq-1}\,(next,\ s)) \rightarrow \text{extensible}\,(s)$

THEOREM: extensible-necc
$(\neg\ ((above < \text{next}\,(above,\ s)) \land \text{prehom-seq-1}\,(\text{next}\,(above,\ s),\ s)))$
$\rightarrow\quad (\neg\ \text{extensible}\,(s))$

DEFINITION:
above-all-aux $(a,\ s,\ n)$
$=$ **if** $n \simeq \mathtt{0}$ **then** $\mathtt{0}$
else above (cons (cons $(a,\ n),\ s)) +$ above-all-aux $(a,\ s,\ n-1)$ **endif**

DEFINITION: above-all $(a,\ s) =$ above-all-aux $(a,\ s,\ \text{BOUND})$

THEOREM: lessp-above-all-aux
$(\mathtt{0} < n) \rightarrow (\text{above-all-aux}\,(a,\ s,\ n) \not< \text{above}\,(\text{cons}\,(\text{cons}\,(a,\ n),\ s)))$

THEOREM: above-all-aux-monotone
$(bound \not< n) \rightarrow (\text{above-all-aux}\,(a,\ s,\ bound) \not< \text{above-all-aux}\,(a,\ s,\ n))$

THEOREM: lessp-above-all-bound
$((\mathtt{0} < n) \land (\text{BOUND} \not< n))$
$\rightarrow\quad (\text{above-all}\,(a,\ s) \not< \text{above}\,(\text{cons}\,(\text{cons}\,(a,\ n),\ s)))$

DEFINITION: next-element $(a,\ s) =$ next (above-all $(a,\ s),\ s)$

DEFINITION: next-color $(a,\ s) =$ p $(a,$ next-element $(a,\ s))$

THEOREM: numberp-p
$\text{p}\,(x,\ y) \in \mathbf{N}$

THEOREM: next-color-bound
$(\mathtt{0} < \text{next-color}\,(a,\ s)) \land (\text{BOUND} \not< \text{next-color}\,(a,\ s))$

EVENT: Disable above-all.

```
;; first of two goals for extensible-cons
```

THEOREM: lessp-next-element
extensible (s)
$\rightarrow$ (above (cons (cons $(a,$ next-color $(a,\ s)),\ s)) <$ next-element $(a,\ s))$

THEOREM: prehom-seq-1-next-element
extensible $(s) \rightarrow$ prehom-seq-1 (next-element $(a,\ s),\ s)$

EVENT: Disable next-element.

```
;; second of two goals for extensible-cons
```

THEOREM: prehom-seq-1-next-element-cons
extensible (s)
$\rightarrow$ prehom-seq-1 (next-element $(a,\ s)$, cons (cons $(a,$ next-color $(a,\ s)),\ s))$

EVENT: Disable next-color.

THEOREM: extensible-cons
extensible $(s) \rightarrow$ extensible (cons (cons $(a,$ next-color $(a,\ s)),\ s))$

DEFINITION:
next-pair (s) = cons (next (caar $(s),\ s)$, next-color (next (caar $(s),\ s),\ s))$

THEOREM: next-pair-extends
extensible $(s) \rightarrow$ extensible (cons (next-pair $(s),\ s))$

DEFINITION: ramsey-seq-p (s) = (extensible $(s) \land$ prehom-seq $(s))$

THEOREM: extensible-next-property
extensible $(s) \rightarrow ((a <$ next $(a,\ s)) \land$ prehom-seq-1 (next $(a,\ s),\ s))$

THEOREM: ramsey-seq-p-extends
ramsey-seq-p $(s) \rightarrow$ ramsey-seq-p (cons (next-pair $(s),\ s))$

THEOREM: extensible-nlistp
$(\neg$ listp $(s)) \rightarrow$ extensible (s)

THEOREM: ramsey-seq-p-nlistp
$(s \simeq \mathbf{nil}) \rightarrow$ ramsey-seq-p (s)

EVENT: Disable ramsey-seq-p.

EVENT: Disable next-pair.

DEFINITION:
ramsey-seq (n)
$=$ **if** $n \simeq 0$ **then nil**
else cons (next-pair (ramsey-seq $(n - 1)$), ramsey-seq $(n - 1)$) **endif**

THEOREM: ramsey-seq-has-ramsey-seq-p
ramsey-seq-p (ramsey-seq (n))

```
;; Now we want to cut down this prehomogenous sequence to one that's
;; homogeneous.  First let's define the flag.
```

DEFINITION:
good-color-p (c)
$\leftrightarrow$ $\forall$ *big* $\exists$ *good-c-index* ((*big* $<$ *good-c-index*)
$\wedge$ ($c =$ cdr (car (ramsey-seq (*good-c-index*)))))

```
;; The following two lemmas are for the benefit of the proof-checker
;; macro-command SK*.
```

THEOREM: good-color-p-suff
((big $(c) <$ *good-c-index*) $\wedge$ ($c =$ cdar (ramsey-seq (*good-c-index*))))
$\rightarrow$ good-color-p (c)

THEOREM: good-color-p-necc
($\neg$ ((*big* $<$ good-c-index (*big*, c))
$\wedge$ ($c =$ cdar (ramsey-seq (good-c-index (*big*, c))))))
$\rightarrow$ ($\neg$ good-color-p (c))

DEFINITION:
good-c-index-wit (*bound*)
$=$ **if** *bound* $\simeq 0$ **then** 1
else big (*bound*) + good-c-index-wit (*bound* $- 1$) **endif**

THEOREM: good-c-index-wit-positive
$0 <$ good-c-index-wit (*bound*)

THEOREM: lessp-big-good-c-index
(($0 < c$) $\wedge$ (*bound* $\not< c$))
$\rightarrow$ ((big $(c) <$ good-c-index-wit (*bound*)) $=$ **t**)

EVENT: Disable good-c-index-wit.

DEFINITION: COLOR $=$ cdar (ramsey-seq (good-c-index-wit (BOUND)))

THEOREM: ramsey-seq-has-colors-in-bounds
$(0 < n)$
$\rightarrow \quad ((0 < \text{cdar}(\text{ramsey-seq}(n))) \wedge (\text{BOUND} \not< \text{cdar}(\text{ramsey-seq}(n))))$

THEOREM: color-in-bounds
$(0 < \text{COLOR}) \wedge (\text{BOUND} \not< \text{COLOR})$

THEOREM: color-is-good
good-color-p (COLOR)

EVENT: Disable color.

EVENT: Disable *1*color.

DEFINITION:
ramsey-index (n)
$=$ **if** $n \simeq 0$ **then** good-c-index (0, COLOR)
else good-c-index (ramsey-index ($n - 1$), COLOR) **endif**

THEOREM: color-properties
$(big < \text{good-c-index}(big, \text{COLOR}))$
$\wedge \quad (\text{cdr}(\text{car}(\text{ramsey-seq}(\text{good-c-index}(big, \text{COLOR})))) = \text{COLOR})$

THEOREM: ramsey-index-increasing
$(i < j) \rightarrow (\text{ramsey-index}(i) < \text{ramsey-index}(j))$

DEFINITION: ramsey (n) = car (car (ramsey-seq (ramsey-index (n))))

```
#| Next goal:
(prove-lemma ramsey-increasing (rewrite)
  (implies (lessp i j)
   (lessp (ramsey i) (ramsey j))))
|#
```

THEOREM: good-c-index-numberp
good-c-index (big, COLOR) $\in \mathbf{N}$

THEOREM: ramsey-index-numberp
ramsey-index (n) $\in \mathbf{N}$

THEOREM: car-next-pair
car (next-pair (s)) = next (caar (s), s)

THEOREM: ramsey-seq-extensible
extensible (ramsey-seq (n))

THEOREM: next-not-zerop
extensible $(s) \rightarrow ((\text{next}\,(a,\ s) \in \mathbf{N}) \land (\text{next}\,(a,\ s) \neq 0))$

THEOREM: ramsey-seq-increasing
$(i < j) \rightarrow ((\text{caar}\,(\text{ramsey-seq}\,(i)) < \text{caar}\,(\text{ramsey-seq}\,(j))) = \mathbf{t})$

THEOREM: ramsey-index-increasing-rewrite
$(i < j) \rightarrow ((\text{ramsey-index}\,(i) < \text{ramsey-index}\,(j)) = \mathbf{t})$

EVENT: Disable ramsey-index-increasing.

THEOREM: ramsey-increasing
$(i < j) \rightarrow (\text{ramsey}\,(i) < \text{ramsey}\,(j))$

```
;; Now we want to show that ramsey is homogeneous for (color):

#|
(prove-lemma ramsey-homogeneous (rewrite)
  (implies (lessp i j)
   (iff (p (ramsey i) (ramsey j))
(color))))
|#
```

DEFINITION:
restn $(n,\ l)$
= **if** listp (l)
 then if $n \simeq 0$ **then** l
  else restn $(n - 1,\ \text{cdr}\,(l))$ **endif**
 else l **endif**

```
;; We've already proved (ramsey-seq-p (ramseq-seq i)).  So, in order
;; to prove the key lemma ramsey-seq-prehom below, we'll use this
;; together with an appropriate fact about restn and prehom seqs.
```

THEOREM: ramsey-seq-restn-length
restn $(n,\ \text{ramsey-seq}\,(n)) = \mathbf{nil}$

THEOREM: prehom-seq-ramsey
prehom-seq (ramsey-seq (n))

DEFINITION:
length (l)
= **if** listp (l) **then** $1 + \text{length}\,(\text{cdr}\,(l))$
 else 0 **endif**

THEOREM: prehom-seq-1-restn
$(\text{prehom-seq-1}\,(a,\ s) \wedge (x < \text{length}\,(s)))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{restn}\,(x,\ s)),\ a) = \text{cdar}\,(\text{restn}\,(x,\ s)))$

THEOREM: prehom-seq-restn
$(\text{prehom-seq}\,(s) \wedge (\mathtt{0} < x) \wedge (x < \text{length}\,(s)))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{restn}\,(x,\ s)),\ \text{caar}\,(s)) = \text{cdar}\,(\text{restn}\,(x,\ s)))$

THEOREM: ramsey-seq-plus
$\text{restn}\,(x,\ \text{ramsey-seq}\,(x + y)) = \text{ramsey-seq}\,(y)$

THEOREM: plus-comm
$(x + y) = (y + x)$

THEOREM: ramsey-seq-plus-commuted
$\text{restn}\,(x,\ \text{ramsey-seq}\,(y + x)) = \text{ramsey-seq}\,(y)$

THEOREM: length-ramsey-seq
$\text{length}\,(\text{ramsey-seq}\,(n)) = \text{fix}\,(n)$

```
;; The lemmas from here to RAMSEY-SEQ-PREHOM were to eliminate the
;; proof-checker hints from that lemma.
```

THEOREM: plus-difference-elim
$((j \in \mathbf{N}) \wedge (j \not< i)) \rightarrow ((i + (j - i)) = j)$

THEOREM: restn-difference-ramsey-seq
$((\mathtt{0} < i) \wedge (i < j))$
$\rightarrow \quad (\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j)) = \text{ramsey-seq}\,(i))$

THEOREM: prehom-seq-restn-commuted
$(\text{prehom-seq}\,(s) \wedge (\mathtt{0} < x) \wedge (x < \text{length}\,(s)))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(s),\ \text{caar}\,(\text{restn}\,(x,\ s))) = \text{cdar}\,(\text{restn}\,(x,\ s)))$

THEOREM: ramsey-seq-prehom-lemma
$((\mathtt{0} < i) \wedge (i < j))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j))),\ \text{caar}\,(\text{ramsey-seq}\,(j)))$
$\qquad = \quad \text{cdar}\,(\text{restn}\,(j - i,\ \text{ramsey-seq}\,(j))))$

THEOREM: ramsey-seq-prehom
$((\mathtt{0} < i) \wedge (i < j))$
$\rightarrow \quad (\text{p}\,(\text{caar}\,(\text{ramsey-seq}\,(i)),\ \text{caar}\,(\text{ramsey-seq}\,(j))) = \text{cdar}\,(\text{ramsey-seq}\,(i)))$

THEOREM: cdar-ramseq-seq-ramsey-index
$\text{cdar}\,(\text{ramsey-seq}\,(\text{ramsey-index}\,(n))) = \text{COLOR}$

THEOREM: lessp-good-c-index
$(big < \text{good-c-index}\,(big,\ \textsc{color})) = \mathbf{t}$

EVENT: Disable color-properties.

THEOREM: good-c-index-non-zero
$(\mathtt{0} < \text{good-c-index}\,(big,\ \textsc{color})) = \mathbf{t}$

THEOREM: ramsey-index-positive
$\mathtt{0} < \text{ramsey-index}\,(n)$

THEOREM: ramsey-seq-hom-lessp
$(i < j) \rightarrow (\text{p}\,(\text{ramsey}\,(i),\ \text{ramsey}\,(j)) = \textsc{color})$

THEOREM: p-num-is-p
$((x \in \mathbf{N}) \wedge (y \in \mathbf{N})) \rightarrow (\text{p-num}\,(x,\ y) = \text{p}\,(x,\ y))$

THEOREM: numberp-ramsey
$\text{caar}\,(\text{ramsey-seq}\,(\text{ramsey-index}\,(n))) \in \mathbf{N}$

THEOREM: ramsey-seq-hom
$((i \in \mathbf{N}) \wedge (j \in \mathbf{N}) \wedge (i \neq j))$
$\rightarrow \quad (\text{p-num}\,(\text{ramsey}\,(i),\ \text{ramsey}\,(j)) = \textsc{color})$

```
;; The above, together with what was already proved, i.e.

;; (prove-lemma ramsey-increasing (rewrite)
;;  (implies (lessp i j)
;;      (lessp (ramsey i) (ramsey j))))

;; and the fact that p-num was arbitrary (and there are no add-axioms)
;; finishes the job.
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