Expected Coverage of Computer Sciences 313K

DRAFT

1. Sentential Calculus (SC)
   1.1. The basics: syntax, semantics, tautological consequence, tautologies, SC formal proofs.
   1.2. Translating ideas to logic and back again
   1.3. Conjuntive Normal Form and Resolution Proofs.

Problems that any passing students should be able to solve:

1. Prove that for all propositions *p*, *q*, and *r*:

$$[p \Rightarrow (q \wedge r)] \Rightarrow [\sim p \vee (q \vee r)]$$

2. Prove that the conclusion $b \wedge c$ **follows from the premises** $a \Rightarrow (c \vee d), b \Rightarrow a, d \Rightarrow c, \text{and}\, b$ **. First convert the premises and the negation of the conclusion into Conjunctive Normal Form, then employ a resolution proof to get a contradiction.**

2. First Order Predicate Calculus (PC):
   2.1. Syntax, semantics,
   2.2. Consequence, valid sentences, simple PC formal proofs.
   2.3. Genuine understanding of quantifiers, particularly when they are nested
   2.4. Translating ideas to logic and back again using quantifiers.

Problems that any passing students should be able to solve:

3. State an assertion that, if true, would falsify each of the following claims:
   a) All zamzows have tenockritus.
   b) Some zamzows have tenockritus.

4. Given the following two axioms: $\forall x\ Px \Rightarrow Qx$ ***and*** $\exists x \sim Qx \wedge \sim Rz$, **prove that** $\exists y \sim Py$

5. Consider the predicates *Wx* (*x* is a woman), *CHILDOFxy* (*y* is a child of *x*), and *Mx* (*x* is a mother).

a) Write a PC formula to describe the fact that all women with children are mothers.

b) Specify a predicate *P* that makes the following true given the universe of people. You must state a nontrivial predicate (i.e., True is not an acceptable answer):

$$\forall x \sim Px \Rightarrow \exists y,z\ CHILDOFyx \wedge CHILDOFyz \wedge x \neq z$$

6. Write the negation of $\exists m\ \exists n\ \forall x\ Pxn \Rightarrow Qxm$.

7. Write PC statements to express each of the following facts. You may use any of the following predicates:

- ***Exy* ≡ *x* has sent an email message to *y*.**
- ***Txy* ≡ *x* has telephoned *y*.**
- ***Cxyr* ≡ *x* has chatted with y in on-line chat room *r*.**
- ***Sxy* ≡ *x* has taken course *y*.**
- ***Oxy* ≡ department *x* offers course *y*.**

Make sure that you state the universe of discourse for each quantified variable you use (e.g. *S = {students in your school}, C = {on-line chat rooms}*.)

a). There are two students in your school who, between them, have emailed or telephoned everyone else in the school.
b). Every student in your school has chatted with at least one other student in at least one on-line chat room.
c). There is a student in your school who has not received an email message from anyone else in the school.

9. Let the universe of discourse for *x* and *y* be the positive integers. Define $GTEyx \equiv (y \geq x)$. Give a counterexample to the following assertion:

$$\forall x\ ((\exists y\ GTEyx) \Rightarrow (\exists y \sim GTEyx))$$

10. Let the universe of discourse for *x*, *y*, and *z* be the set of students at your school. Let *Fxy* be true iff *x* and *y* are friends. Translate the following statement into English:

$$\exists x\ \forall y\ \forall z\ (Fxy \wedge Fxz \wedge y \neq z) \Rightarrow \sim Fyz$$

3. Sets:
   3.1. Definition of set and of the basic set operations
   3.2. Translating set definitions to formal statements and back again
   3.3. Use of Venn diagrams to visualize set operations
   3.4. Cardinality (of finite sets). The Principle of Inclusion and Exclusion.
   3.5. Theorems about sets and their operations (union, intersection, etc.)
   3.6. Cartesian products of sets
   3.7. Proving theorems about sets

Problems that any passing students should be able to solve:

11. What are these sets? Write them using braces, commas, numerals, … (for infinite sets), and ∅ only. *N* is the set of natural numbers. ($\mathcal{P}(S)$ is the power set of *S*.)

a) $(\{1, 2, 5\} - \{5, 7, 9\}) \cup (\{5, 7, 9\} - \{1, 2, 5\})$
b) $\{1\} \cup \{\varnothing\} \cup \varnothing$
c) $\mathcal{P}(\{7, 8, 9\}) - \mathcal{P}(\{7, 9\})$
d) $\{x : x$ is an integer and $x^2 = 2\}$
e) $\{x : \exists y \in N$ where $x = y^2\}$
f) $\{1\} \times \{1, 2\} \times \{1, 2, 3\}$
g) $\varnothing \times \{1, 2\}$
h) $\mathcal{P}(\{1, 2\}) \times \{1, 2\}$

12. What is the cardinality of each of the following sets? Justify your answer.
a) $S = \{\varnothing, \{\varnothing\}\}$
b) $S = \mathcal{P}(\{a, b, c\})$
c) $S = \{a, b, c\} \times \{1, 2, 3, 4\}$

13. Let *N* be the set of nonnegative integers. Let $S = \{x \in Z: \exists y \in N$ where $x = 2y\}$ and $T = \{x \in Z: \exists y \in N$ where $x = 2^y\}$.
a). Define $W = S - T$. Describe *W* in English. List any five consecutive elements of *W*.
b). Define $X = T - S$. Describe *X* in English?

14. Prove the following for all sets, *A*, *B*, *C*, and *D*. Do this either syntactically, using the set identities, or semantically, by writing logical assertions that must be true of the elements in the two sets.
a). $(A \cap B) - C \subseteq (A \cup D) \cap B$.
b). $(A - B) - C = A - (B \cup C)$

15. Clearly describe the difference between $\emptyset$ and $\{\emptyset\}$. What is the cardinality of each of these sets?

16. Let $S = \{a, b, c, d\}$. Let $X = \{A \subseteq S : a \in A \rightarrow b \notin A\}$. List all elements of X.

17. Suppose in a class, 26 students got an A on Exam 1 and 21 got an A on Exam 2. If 30 got an A on at least one of the two exams, how many got A's on both exams?

18. Let *Z* denote the set of integers. Using the # vocabulary (see below), describe the set of even integers greater than 5.

19. Given a set *X* of subsets of a set *S*, define, using the # vocabulary, the set of elements of *X* that have cardinality equal to 2 (i.e., contain exactly two objects from *S*).

20. Given a set *X* of subsets of a set *S*, define, using the # vocabulary, the set of elements from *S* that appear in exactly one element of *X*.

21. Let *A, B* be two sets. If $\mathcal{P}(A) = \mathcal{P}(B)$ must $A = B$? Prove your answer.

4. Relations
   4.1. Definition of a relation
   4.2. Binary relations
      4.2.1. Properties: reflexive, symmetric, transitive, antisymmetric.
      4.2.2. Partial and total orderings of sets defined by a relation.
      4.2.3. Equivalence relations and partitions.
      4.2.4. Closures of relations.

Problems that any passing students should be able to solve:

22. For each of the following sets, state whether or not it is a partition of {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
- **a). {{0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}**
- **b). {$\emptyset$, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}**
- **c). {{1, 2}, {3, 4}, {5, 6}, {7, 8}, {9, 10}}**
- **d). {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}}**

23. For each of the following relations, state which of these properties hold:reflexivity, symmetry, transitivity, and antisymmetry.
- **a). = defined on strings**
- **b). $\neq$ defined on strings**
- **c). < defined on *N* (the natural numbers)**
- **d). subset of defined on the power set of *N***

24. For each of the following relations *R*, over some domain *D*, compute the reflexive, symmetric, transitive closure *R′*. Try to think of a simple descriptive name for the new relation *R′*. Since *R′* must be an equivalence relation, describe the partition that *R* induces on *D*.

a). Let *D* be the set of 50 states in the US. $\forall xy, xRy$ iff *x* shares a boundary with *y*.

b). Let *D* be the natural numbers. $\forall xy, xRy$ iff $y = x+3$.

c). Let *D* be the set of strings containing no symbol except *a*. $\forall xy, xRy$ iff $y = xa$. (i.e., if *y* equals *x* concatenated with *a*).

25. Let *A* be a set of people. Let *F* be the friendship relation on *A*. In other words, *xFy* iff *x* is friends with *y*. We will say that *A* is a "friendly bunch of people" if everyone in *A* is friends with at least as many people as they are not friends with. Using the # vocabulary, define this predicate formally. Write a logical expression that describes the set of elements *x* of a set *S* partially ordered by ≥, such that there are exactly two elements in *S*, other than *x*, that are greater than or equal to *x*.

26. For each of the following relations, state whether it is a partial order (that is not also total), a total order, or neither. Justify your answer.

a). *DivisibleBy*, defined on the natural numbers. $(x, y) \in DivisibleBy$ iff *x* is evenly divisible by *y*. So, for example, $(9, 3) \in DivisibleBy$ but $(9, 4) \notin DivisibleBy$.

b). *LessThanOrEqual* defined on ordered pairs of natural numbers. $(a, b) \leq (x, y)$ iff $a \leq x$ or ($a = x$ and $b \leq y$). For example, $(1,2) \leq (2,1)$ and $(1,2) \leq (1,3)$.

5. Functions:
   - 5.1. Definition
   - 5.2. Properties of functions (one-to-on, onto, bijection)
   - 5.3. Optional: pigeonhole principle
   - 5.4. Inverses
   - 5.5. Identities
   - 5.6. Composition

Problems that any passing students should be able to solve:

27. Let $R = \{(a, b), (a, c), (c, d), (a, a), (b, a)\}$. What is R ° R, the composition of R with itself? What is R^{-1}, the inverse of *R*? Is *R*, $R \circ R$, or R^{-1} a function?

28. For each of the following functions, state whether or not it is (i) one-to-one, and (ii) onto. Justify your answers. Let *Z* be the set of integers, *N* be the set of nonnegative integers, and *P* be the set of positive integers.

a). $F: Z \to N$, where

b). $F(x) = 1 + x^2$

c). $G: N \to N$, where

d). $G(x) = x + 1$

e). $H: Z \to N$, where

f). $H(x) = x^2$

g). $K: Z \to Z$ where $K(x) = -x$

h). $+: P \times P \to P$, where $+(a, b) = a + b$ (In other words, simply addition defined on the positive integers)

i). $X : B \times B \to B$, where *B* is the set {True, False} and $X(a, b)$ = the exclusive or of a and b

29. Let D be the set of people. For each of the following relations answer these questions: (i) Is it reflexive? (ii) Is it symmetric? (iii) Is it transitive? (iv) Is it a function? (v) If it is a function, is it one-to-one? (vi) If it is a function, is it onto?

a). $\forall x,y \in D\ xCy$ iff *y* is a child of *x*

b). $\forall x,y \in D\ xMy$ iff *y* is the mother of *x*

c). $\forall x,y \in D\ xPy$ iff *y* is a parent of *x*

d). $\forall x,y \in D\ xRy$ iff *y* is a blood relative of *x*

30. Define $(f \circ g)(x) \equiv f(g(x))$. Define the following functions on the integers:

$d(x) = 2x$

$v(x) = -x$

$a(x, y) = x + y$

$f(x) = a((d \circ v)(x), 10)$

a). What is $f(3)$?

b). Is *f* one-to-one? Explain.

c). Is *f* onto? Explain.

d). Is the inverse of *f* a function? If it is, what is its domain? What is its range? Explain.

31. Using the # vocabulary, write a logical formula to express the property that a function $f: A \to B$ is one-to-one.

32. Using the # vocabulary, write a logical formula to express the property that a function $f: A \to B$ is onto.

33. Are the following sets closed under the following operations? If not, what are the respective closures?

a). The odd integers under multiplication.
b). The positive integers under division.
c). The negative integers under subtraction.
d). The negative integers under multiplication.
e). The odd length strings under concatenation.

6. Recursive Definitions and Mathematical Induction:
   6.1. The natural number system; definition by recursion; definition of plus and times for natural numbers; proofs of basic properties of plus and times; ordering the natural numbers.
   6.2. Proof by induction
      6.2.1. For theorems about the natural numbers
      6.2.2. For other things, e.g., sets

Problems that any passing students should be able to solve:

34.Prove that for all nonnegative integers n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$**. (Recall that an empty summation has the value zero.)**

35. For n ≥ 2, let A1, A2, …, An be a collection of n sets. Using induction, prove the n-fold generalization of the DeMorgan Law:

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$$

(You may assume that $\bigcup_{i=1}^{n+1} B_i = (\bigcup_{i=1}^{n} B_i) \cup B_{n+1}$ **and** $\bigcap_{i=1}^{n+1} B_i = (\bigcap_{i=1}^{n} B_i) \cap B_{n+1}$ **for any collection of n+1 sets.)**

Allowable Notation for Problems with (#)

You may use any of the following symbols in your answer:

- Numbers
- Letters, to represent objects (*e.g.*, sets or elements of sets)
- Standard logical symbols, including $\wedge$, $\vee$, $\sim$, $\Rightarrow$, $\equiv$.
- Standard notation for sets, including {, }, : (such that), $\in$, $\notin$, $\subseteq$, $\cup$, $\cap$, $\varnothing$, and $\overline{}$ (complement)
- $|A|$ for the cardinality of A
- $A \times B$ for the cross product of A and B (and its extension to a k-way cross product)
- (and), both as delimiters and to indicate an ordered tuple (*e.g.*, (a, b, c))
- 2^S or $\mathcal{P}(S)$ to indicate the power set of S
- Comparison operators, including $\leq$, $\geq$, $=$, $\neq$
- Arithmetic operators, including +, -, *, /
- Quantifiers: $\forall$ and $\exists$
- Predicates expressed as Px for P holds of x