Asymptotic Dominance Theory

- **Definition 1**: Given the functions $f:\mathrm{N}\to R$ and $g:\mathrm{N}\to R$, f is *asymptotically dominated by* g if there exist non-negative constants M and N such that for all $n \geq N$, $|f(n)| \leq M|g(n)|$. This is denoted by $f = \mathrm{O}(g)$.

- **Definition 2**: Given the functions $f: N \to R$ and $g: N \to R$, $f = o(g)$ if for every positive ε, there exists a non-negative constant N such that for all $n \geq N$, $|f(n)| \leq \varepsilon |g(n)|$.

Theorem 1: If $f = \mathrm{O}(g)$, then for any constant *s*, $sf = \mathrm{O}(g)$.

Theorem 2: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 + f_2 = \mathrm{O}(|g_1| + |g_2|)$.

Corollary 2.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(\sum_{i=1}^{k} |g_i|)$.

Theorem 3: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 + f_2 = \mathrm{O}(\max\{|g_1|, |g_2|\})$.

Corollary 3.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(\max_{i=1,\ldots,k} |g_i|)$.

Corollary 3.2: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g)$, then $\sum_{i=1}^{k} f_i = \mathrm{O}(g)$.

Theorem 4: If $f_1 = \mathrm{O}(g_1)$ and $f_2 = \mathrm{O}(g_2)$, then $f_1 \cdot f_2 = \mathrm{O}(g_1 \cdot g_2)$.

Corollary 4.1: If for $i = 1, 2, \ldots, k$, $f_i = \mathrm{O}(g_i)$, then $\prod_{i=1}^{k} f_i = \mathrm{O}(\prod_{i=1}^{k} g_i)$.

Theorem 5: If $f_1 = \mathrm{O}(g_1)$, $g_2 = \mathrm{O}(f_2)$, and g_2 has no zeros. then $f_1 / f_2 = \mathrm{O}(g_1 / g_2)$.

Theorem 6: If $0 \leq a \leq b$, then $n^a = \mathrm{O}(n^b)$

Theorem 7: If $0 \leq a < b$, then $n^a = o(n^b)$