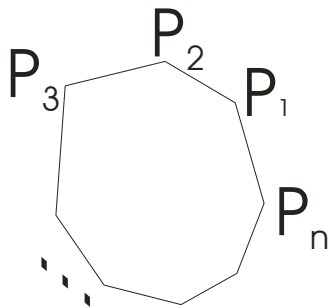


**Examination 1 Solutions**  
**CS 336**

1. [5] For  $n \geq 3$ , how many diagonals does a convex polygon with  $n$  extreme points have? (Consider a convex polygon given by extreme points  $\langle P_1, P_2, \dots, P_n \rangle$  in counterclockwise order. A “diagonal” is a line segment connecting two **non-adjacent** extreme points.)



For each of the  $n$  extreme points there are  $n-3$  distinct extreme points that non-adjacent. This would yield  $n(n-3)$  endpoints of the diagonals. Since each diagonal has two endpoints, there are  $\frac{n(n-3)}{2}$  diagonals of a convex polygon with  $n$  extreme points.

2. a. [10] Present a combinatorial argument that for all  $n \geq 1$ :

$$(2n-1) \cdot (2n-3) \cdots 3 \cdot 1 = \frac{(2n)!}{n!2^n}$$

Consider the set of all partitions of a set of cardinality  $2n$  into  $n$  pairs. For the left side, begin with any permutation of the  $2n$  elements. The first element on the permutation is in some pair and there are  $2n-1$  choices for its pair-mate. Removing these two from the permutation, the next element permutation is also in some pair and there are  $2n-3$  choices for its pair-mate. The process continues until there are just two elements left in the permutation, and they form the last pair. This yields  $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$  different such partitions. Now consider the right hand side. There are  $(2n)!$  different permutations of the of the  $2n$  elements. Pair the first element with the second, the third with the fourth, etc. This yields a partition into  $n$  pairs. However, the order among the  $n$  pairs is irrelevant to the partition and thus for every array of pairs there are  $2^n$  different permutations. Lastly, the order among the pairs, is also irrelevant, so a set of pairs could be arranged in  $n!$  different orders. Thus the number of partitions into pairs that ignores order within and among pairs is  $\frac{(2n)!}{n!2^n}$  and this must equal  $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$ .

b. [10] Present a combinatorial argument that for all nonnegative integers  $k$  and  $n$  satisfying  $k \leq n-2$

$$\binom{n+2}{k} = \binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$$

Let set  $A$  have cardinality  $n$  and  $b$  and  $c$  be distinct elements not contained in  $A$ . Consider the subsets of  $A \cup \{b\} \cup \{c\}$  of cardinality  $k$ . For the left hand side, we recognize that  $A \cup \{b\} \cup \{c\}$  has cardinality  $n+2$ , so there are  $\binom{n+2}{k}$  such subsets.

Alternatively, consider that a subset wither has all  $k$  elements coming from  $A$ , exactly  $k-1$  elements coming from  $A$ , or  $A$ , exactly  $k-2$  elements coming from  $A$ . If all  $k$  elements come from  $A$ , there are  $\binom{n}{k}$ . If exactly  $k-1$  elements come

from  $A$ , there are  $\binom{n}{k-1}$  ways to select those elements and then two choices,  $b$  or  $c$ , to complete the subset. If exactly  $k-2$  elements come from  $A$ , there

are  $\binom{n}{k-2}$  ways to select those elements and then both  $b$  and  $c$  must be selected to

complete the subset. The total is  $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$  and this must

equal  $\binom{n+2}{k}$ .

3. [15] How many partitions are there of a set of 45 elements into a subset of cardinality 3, six subsets of cardinality 4, and three subsets of cardinality 6?

There are  $\binom{45}{3}$  ways to select the elements for the subset of cardinality 3. Removing

those leaves 42 elements, and there are  $\binom{42}{4 \ 4 \ 4 \ 4 \ 4 \ 4}$  ways to select the

elements for the six subsets of cardinality 4. However, there is no order amongst these subsets and there are actually  $6!$  ways to reorder the subsets, so this has been over counted by a factor of  $6!$ . Finally, removing those leaves 18 elements, and there

are  $\binom{18}{6 \ 6 \ 6}$  ways to select the elements for the three subsets of cardinality 6.

However, there is no order amongst these subsets and there are  $3!$  ways to reorder the subsets, so this has been over counted by a factor of  $3!$ . The number of such

partitions is then  $\frac{\binom{45}{3} \binom{42}{4 \ 4 \ 4 \ 4 \ 4} \binom{18}{6 \ 6 \ 6}}{6!3!}$ . This can also be written

$$\frac{\binom{45}{3 \ 4 \ 4 \ 4 \ 4 \ 4 \ 6 \ 6 \ 6}}{6!3!}.$$

4. [15] For  $n \geq 1$ , what is the value of  $\sum_{i_1=0}^n \left( \sum_{i_2=0}^{n-i_1} \left( \sum_{i_3=0}^{n-i_1-i_2} 1 \right) \right)$ ? Present a combinatorial argument: determine the value of the expression, then defend it by establishing a model and counting it. (Hint: Define  $i_4 = n - i_1 - i_2 - i_3$  and then think about putting balls into bins.)

The expression adds one for every ordered triple  $\langle i_1, i_2, i_3 \rangle$  of non-negative integers so that  $i_1 + i_2 + i_3 \leq n$ . Letting  $i_4 = n - i_1 - i_2 - i_3$ , this is the same as adding one for every ordered four-tuple  $\langle i_1, i_2, i_3, i_4 \rangle$  of non-negative integers so that  $i_1 + i_2 + i_3 + i_4 = n$ . That however, is the number of ways of tossing  $n$  balls into four bins which is  $\binom{n+3}{3}$ .

5. [10] Consider strings of four  $a$ s and four  $b$ s. Assume all such strings are equally likely. What is the probability that two or more  $a$ s precede all of the  $b$ s.

A string has length 8 and is determined by the 4 positions for the  $a$ s, thus there are  $\binom{8}{4}$  equally likely strings. Either exactly two  $a$ s precede all of the  $b$ s, exactly three  $a$ s precede all of the  $b$ s, or all four  $a$ s precede all of the  $b$ s. If exactly two  $a$ s precede all of the  $b$ s, the string begins  $aab$  and there are 5 remaining positions to contain the 2  $a$ s. Thus  $\binom{5}{2}$  such strings. If exactly three  $a$ s precede all of the  $b$ s, the string begins  $aaab$  and there are 4 remaining positions to contain the last  $a$ . Thus there are  $\binom{4}{1}$  such strings. Lastly there is only one string in which all of the  $a$ s precede all of the  $b$ s. In total there are  $\binom{5}{2} + \binom{4}{1} + 1 = 15$  strings in which two or more  $a$ s precede all of the  $b$ s. The probability of having such a string is  $\frac{15}{\binom{8}{4}}$

**6. [10]** Consider rolls of a pair of six-sided dice assuming all possible order pairs of outcomes are equally likely. What is the probability that the sum of values shown on the dice is 8 given that either of the dice shows a 2?

There are  $6^2 = 36$  equally likely rolls of the dice. If the first die shows a 2, there are 6 different values for the second die (one of which is 2). If the second die shows a 2, there are 6 different values for the first die (one of which is 2). Thus,  $6+6-1=11$  different rolls have at least one die with a value of 2. Of these, there are the only way to get a total of 8 is with (2,6) or (6,2). Thus, the probability that the sum of values shown on the dice is 8 given that either of the dice shows a 2 is  $\frac{2}{11}$ .