Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set of reciprocals of positive integers (i.e. $\{\frac{1}{1},\frac{1}{2},\frac{1}{3},...\} = \{\frac{1}{k} \mid k \in \mathbb{Z} \wedge k \geq 1\}$ is infinite**.**

Let R = set of set of reciprocals of positive integers. For $r \in R$, define $f : R \to R$ by $f(r) = \frac{r}{2}$ (i.e. if $r = \frac{1}{k}$ then $f(r) = \frac{1}{2k}$). For $r,s \in R$, if $r \neq s$, then $f(r) = \frac{r}{2} \neq \frac{s}{2} = f(s)$, so f is one-to-one. Since for all $r \in R$ $r \leq 1$, so $f(r) \leq \frac{1}{2}$, and thus there is no $r \in R$ such that $f(r) = \frac{1}{1}$. We have then that f maps R into $R \sim \{1\}$, which is a proper subset of R, and by Definition 2', R is infinite.

2. [20] Let $P = \{$infinitely long bit strings containing exactly ten zeros$\}$. Is the set P finite, countably infinite, or uncountably infinite? Prove your claim.

The set is countably infinite. For $n \in \mathbb{N}$, let P_n be the set of infinitely long bit strings having exactly ten zeros and all the zeros occurring in the first n positions. Each set P_n is finite (in fact, having cardinality $\binom{n}{10}$ if $n \geq 10$ and zero if $n < 10$) and $P = \bigcup_{n \in \mathbb{N}} P_n$ .Since P is the countably infinite union of finite sets, by Theorem 9, it is countable. Finally consider the mapping $f : \mathbb{N} \to P$ defined by $f(n) = <0000000001...101....>$ where there are n ones between the ninth and tenth zeros. This function is clearly one-to-one since if $n \neq m$, $f(n)$ has a zero in position $10+n$ and $f(m)$ has a one in position $10+n$. By Theorem 4, P is infinite, hence countably infinite.

3. [20] Is the set of circles in the plane finite, countably infinite, or uncountably infinite**?** Prove your claim.

The set is uncountably infinite. Let $\mathbb{C}$ denote the set of circles in the plane and consider $f : [0,1] \to \mathbb{C}$ defined by $f(x)$ is the circle or radius 1 with center $(x,0)$. This function is one-to-one since if x and y are elements of $[0,1]$ and $x < y$, then the circle $f(x)$ contains the point $(x-1,0)$ but the circle $f(y)$ does not contain that point since the distance from $(x-1,0)$ to $(y,0)$ is $y-x+1$ which is greater than Thus, f is one-to-one and since the interval $[0,1]$ is uncountably infinite, by Theorem 11, the set $\mathbb{C}$ is uncountably infinite.

4. [20] Using no other asymptotic dominance theory than definitions, prove that $6n^{7/8}+5n^{3/2}=\mathrm{O}(n^2)$.

Let $M=11$ and $N=1$. For $n \geq N$, we have $n^{7/8} \leq n^{3/2} \leq n^2$, so $|6n^{7/8}+5n^{3/2}|=6n^{7/8}+5n^{3/2} \leq 6n^2+5n^2=11n^2=M|n^2|$. Therefore, $6n^{7/8}+5n^{3/2}=\mathrm{O}(n^2)$.

5. [20] Employing induction prove that for $k \geq 1$, if for $i=1,2,\ldots,k$, $f_i=\mathrm{O}(f_{i+1})$, then $f_1=\mathrm{O}(f_{k+1})$.

For k=1, we have $f_1=\mathrm{O}(f_2)$ thus $f_1=\mathrm{O}(f_2)=\mathrm{O}(f_{k+1})$. Let us assume the result is true for some $k \geq 1$, and attempt to prove that if for $i=1,2,\ldots,k+1$, $f_i=\mathrm{O}(f_{i+1})$, then $f_1=\mathrm{O}(f_{k+2})$. By the inductive hypothesis we have $f_1=\mathrm{O}(f_{k+1})$ and we also know $f_{k+1}=\mathrm{O}(f_{k+2})$. By definition, there exist $M,\bar{M},N,$ and $\bar{N}$, so that for $n \geq N, |f_1(n)| \leq M|f_{k+1}(n)|$ and .for $n \geq \bar{N}, |f_{k+1}(n)| \leq \bar{M}|f_{k+2}(n)|$. Thus for $n \geq \bar{\bar{N}}=\max\{N,\bar{N}\}, |f_1(n)| \leq M|f_{k+1}(n)| \leq M\bar{M}|f_{k+2}(n)|$, so $f_1=\mathrm{O}(f_{k+2})$.

6. [20] Prove that $2^n=o(n!)$. (Hint: $\prod_{i=1}^{n}\frac{2}{i}=\prod_{i=1}^{3}\frac{2}{i}\cdot\prod_{i=4}^{n}\frac{2}{i}=\frac{4}{3}\prod_{i=4}^{n}\frac{2}{i}$ and $\frac{2}{i} \leq \frac{1}{2}$ for $i \geq 4$.)

Given any $\varepsilon>0$, let $N=\max\{1,\log_2\frac{32}{3\varepsilon}\}$. Notice then for $n \geq N$, we have

$2^n \geq \frac{32}{3\varepsilon}=\frac{4}{3}\frac{8}{\varepsilon}$, so

$$\varepsilon \geq \frac{4}{3}8\left(\frac{1}{2}\right)^n=\frac{4}{3}\left(\frac{1}{2}\right)^{n-3}=\frac{4}{3}\prod_{i=4}^{n}\frac{1}{2} \geq \frac{4}{3}\prod_{i=4}^{n}\frac{2}{i}=\prod_{i=1}^{n}\frac{2}{i}=\frac{\prod_{i=1}^{n}2}{\prod_{i=1}^{n}i}=\frac{2^n}{n!}.$$

and $|2^n|=2^n \leq \varepsilon n!=\varepsilon|n!|$. Therefore, $2^n=o(n!)$.