CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
comments in boxes

1. a. [5] How many distinct rearrangements are there of the string $< 0000111111 >$?

b. [5] For $n \geq 0$, how many ordered triples $< i, j, k >$ of nonnegative integers satisfy $i + j + k = n$? (Hint: Consider balls and bins.)

2. [10] For $n \geq 1$, how many permutations of 1, 2, 3, ..., *n* are there in which the odd positions have only odd numbers and the even positions have only even numbers? For example, for *n* = 5, the permutations <1, 4, 3, 2, 5> and <3, 2, 1, 4, 5> are allowed but <1, 3, 2, 4, 5> is not. (Hint: You may want to do a separate analysis for even *n* and odd *n*.).

3. a. [10] Using a combinatorial argument, prove that for $n \geq 1$, $m \geq 1$ and $0 \leq l \leq m$:

$$\sum_{k=0}^{n} \binom{n}{k} l^k (m-l)^{n-k} = m^n$$

b. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{n-k} = n \binom{2n-1}{n-1}$$

(Hint: Let *A* and *B* be disjoint sets of cardinality *n*. Consider pairs <*C*, *a*> where $C \subseteq A \cup B$, *C* has cardinality *n*, and $a \in C \cap A$.)

4. a. [5] For $n \geq 5$, what is the probability that a string of *n* zeros and ones has exactly 5 ones. (You may assume all strings of *n* zeros and ones are equally probable.)

b. [10] For $n \geq 5$, what is the probability that a string of *n* zeros and ones has exactly 5 ones given that it has at least 4 ones. (You may assume all strings of *n* zeros and ones are equally probable.)

5. [10] Using definition 2' (and no cardinality theorems) show that the set of non-negative powers of two (i.e., {1, 2, 4, 8, ...}) is infinite.

6. a. [10] Suppose the set A is **uncountably** infinite, the set B is **countably** infinite and $C = A \sim B = \{a \mid a \in A \text{ and } a \notin B\}$. Prove or disprove (with a simple counter example):

C is **uncountably** infinite.

b. [5] Prove that the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ is uncountably infinite.

7. [10] Show that if $f(n) = 12n + 3$ and $g(n) = n^2$, then $f = O(g)$.

8. [10] . Prove that if $0 \leq a < b$, then $n^a = o(n^b)$

9. [10] Prove correct with respect to precondition that *a*, *b*, and *c* are defined and postcondition $(\max = a \vee \max = b \vee \max = c) \wedge (\max \geq a) \wedge (\max \geq b) \wedge (\max \geq c)$:

```
max := a
if b > a then
        {if b > c then
                max : = b
        else
                max := c}
else
        {if c> a then
                max : = c}
```

10. a. [10] Prove the following code is partially correct with respect to precondition "$n \geq 0 \wedge q \neq 0$" and postcondition "$p = q^n$" (assume k, p, n, and q are integer variables.):

```
        k := 1
        p := 1
        while k≤n do
                p := p*q
                k := k+1
        endwhile
```

Be explicit about your loop invariant.

...b. [5] Prove that the loop terminates.

11. [10] Determine the weakest precondition with respect to the postcondition "$even(x)$" for the following code (assume *z*, *y*, and *x* are integer variables and that y is defined):

```
y := 5
z := x+y
if odd(x) then
        x := x-3
else
        x := 2-x
endif
```

12. a. [10] Determine the weakest precondition with respect to the postcondition "z=6" for the following (assume *z*, *y*, and *x* are integer variables):

```
x := 2
z := x-y
if y>0 then
        z := z+1
else
        z := 0
```

b. [5] Determine the weakest precondition with respect to the postcondition "$x \leq y$" for the following (assume *y*, and *x* are integer variables and are defined):

```
if x>y then x := y
```