Final Examination

CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
comments in boxes

1. [10] For $n \geq 1$, how many permutations of 1, 2, 3, ..., *n* are there in which the odd positions have only odd numbers and the even positions have only even numbers? For example, for *n* = 5, the permutations <1, 4, 3, 2, 5> and <3, 2, 1, 4, 5> are allowed but <1, 3, 2, 4, 5> is not. (Hint: You may want to do a separate analysis for even *n* and odd *n*.).

2. [10] A multiset is a generalization of the concept of set and allows multiple copies of its elements. (For example, the multiset $\{a, a, b\}$ is distinct from the multiset $\{a, b, b\}$ since the first has two copies of a and only one copy of b yet the second has two copies of b and only one copy of a.) The cardinality of a multiset is the total number of elements counting copies (thus the cardinality of multisets $\{a, a, b\}$ and $\{a, b, b\}$ is three). Assuming $n \geq 1$ and $m \geq 0$, using elements from the set $\{a_1, \ldots, a_n\}$, how many multisets are there of cardinality m?

3. a. **[10]** Using a combinatorial argument, prove that for $k \geq m \geq 1$:

$$\binom{k+1}{m} = \binom{k}{m-1} + \binom{k}{m}.$$

b. **[10]** Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{n-k} = n \binom{2n-1}{n-1}$$

(Hint: Let *A* and *B* be disjoint sets of cardinality *n*. Consider pairs <*C*, *a*> where $C \subseteq A \cup B$, *C* has cardinality *n*, and $a \in C \cap A$.)

4. a. **[10]** Given a set $A = \{a_1, a_2, \ldots, a_n\}$ with $n \geq 3$, what is the probability that a subset of size *k* has a_1 and a_2? (You may assume $k \geq 2$ and all subsets of size *k* are equally probable.)

. b. **[10]** Given a set $A = \{a_1, a_2, \ldots, a_n\}$ with $n \geq 3$, what is the probability that a subset of size *k* has a_1 and a_2 given that it has at least one of a_1, a_2, or a_3? (You may assume $k \geq 2$ and all subsets of size k are equally probable.)

5. [10] Let $\mathbb{N}$ denote the set of natural numbers and $A = \{a, b, c\}$. Using only definition 2' and no cardinality theorems, prove that $\mathbb{N} \times A$ is infinite.

6. [10] Consider this theorem (that relies upon the Axiom of Choice):

If $f : A \xrightarrow[onto]{} B$, *then there exists a subset* $\hat{A}$ *of* A *such that* $f : \hat{A} \xrightarrow[onto]{1-1} B$.

Use this theorem to prove: If $f : A \xrightarrow[onto]{} B$, and B is infinite then A is infinite.

7. [10] We define $f = \Theta(g)$ if and only if $f = O(g)$ and $g = O(f)$. Prove that $n + \frac{1}{n} = \Theta(17n + 12)$.

8. [10] Prove that if $f_1 = O(g_1)$ and $f_2 = o(g_2)$, then $f_1 \cdot f_2 = o(g_1 \cdot g_2)$.

9. [10] Using Hoare Axioms prove correct with respect to precondition "true" and post-condition $c \leq b \leq a$: (Assume *a*, *b*, *c*, and *t* are integer variables and that *a*, *b*, and *c* are defined.)

```
if (a < b) then
        t := a
        a := b
        b := t
endif
if (b < c) then
        t := b
        b := c
        c := t
        if (a < b) then
                t := a
                a := b
                b := t
        endif
endif
```

10. [10] Prove the following code is partially correct with respect to precondition "$n \geq 0$" and postcondition "$p = a^n$" (assume p, i, and a are integer variables and that a is non-zero.):

Be explicit about your loop invariant: I =

```
p := 1

i := 0

while i ≤ n do

    p := p*a

    i := i+1

endwhile
```

11. a. [10] Determine the weakest precondition with respect to the postcondition "$w > 0$" for the following (assume w, z, y, and x are integer variables and that y and z are defined):

```
x := y
y := x
x := z
y := x
w := x+y+z
```

b. [5] For the same piece of code, determine the weakest precondition with respect to the postcondition "$wy = 12$"

12. [10] Determine the weakest precondition with respect to the postcondition "$y = 1$" for the following code (assume *z*, *y*, and *x* are integer variables and that x and z are defined):

```
if x<3 then
    y := z
    if y<z then
        y := 2*y
    endif
else
    y := z-y
endif
```