Name ______________________________

Final Examination

CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
comments in boxes

1. [10] For $n \geq 7$, how many subsets of size 7 from $\{a_1, a_2, \ldots, a_n\}$ are there that either contain both a_1 and a_2 or both a_3 and a_4 (or all four)?

2. [10] Given $n \geq r \geq 1$, in how many ways can n identical balls be placed into r distinct bins such that each bin contains at least one ball? (Hint: Consider dot diagrams.)

3. a. [10] Using a combinatorial argument, prove that for $n \geq 2$ and $m \geq 2$:

$$\binom{n+m}{2} = n \cdot m + \binom{n}{2} + \binom{m}{2}$$

b. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

(Hint: Let A be a set of cardinality n. Consider pairs $\langle B, a \rangle$ where $B \subseteq A$ and $a \in A \sim B$.)

4. a. [10] For $n \geq 5$, consider strings of length n using elements of $\{1,2,3\}$. Assume all such strings are equally likely. What is the probability that a string has exactly two 3's?

b. [5] What is the probability that such a string has exactly three 2's given that it has exactly two 3's?

5. [10] Using definition 2' (and no cardinality theorems) prove that the set of all integral multiples of 17 (i.e., $\{17k \mid k \in \mathrm{Z}\}$ is infinite.

6. [10] Let $S = \{f \mid f : \mathrm{N} \to \{0,1\}\}$ (i.e., the set of functions mapping the natural numbers into $\{0,1\}$). Is S finite, countably infinite, or uncountably infinite? State then prove your assertion.

7. [10] Prove that $n^3 + n^2 + n = o(n^4)$.

8. a. [10] .Recalling that $\log_a x = \log_b x \cdot \log_a b$, prove that for all $a, b > 1$, if $f = O(\log_a n)$ then $f = O(\log_b n)$.

b. [10]. Consider the following assertion:

For all $a, b > 1$, if $f = O(a^n)$ then $f = O(b^n)$.

Using a simple example, prove that this assertion is false. Notice your example function must be $O(a^n)$ but not $O(b^n)$.

9. [10] Prove the following code is partially correct with respect to precondition "$n \geq 1$" and postcondition "$m = \max\{a_1, a_2, \ldots, a_n\}$":

```
m :=a[1]
i := 2
while i≤n do
        if m<a[i] then
                m := a[i]
        endif
        i := i+1
endwhile
```

You may assume that m, i, n, and the array a are integer variables. You may also assume that the array components $a_1, a_2, \ldots a_n$ are defined. Finally you may use the following axioms:

$$a_i \leq \max\{a_1, \ldots, a_{i-1}\} \Rightarrow \max\{a_1, \ldots, a_i\} = \max\{a_1, \ldots, a_{i-1}\})$$

$$a_i > \max\{a_1, \ldots, a_{i-1}\} \Rightarrow \max\{a_1, \ldots, a_i\} = a_i$$

10. a. [10] Prove the following code is partially correct with respect to precondition "$m \geq 1$ and $n \geq 1$" and postcondition "$b = \binom{m}{n}$" (assume b, m, n, and k are integer variables.):

```
b :=m
k := 2
while k≤n do
        b := (b*(m-k+1)/k
        k := k+1
endwhile
```

You may use the following axiom:

$$\text{For } m \geq 1 \text{ and } k \geq 2, \binom{m}{k-1} \cdot \frac{m-k+1}{k} = \binom{m}{k}.$$

…b. [5] Prove that the loop terminates.

11. [10] Determine the weakest precondition with respect to the postcondition "z=10" for the following code (assume z, y, and x are integer variables and that y is defined):

```
x := 3
z := x-y
if y<0 then
        z := 0
else
        z := 2*z
endif
```

12. [10] Determine the weakest precondition with respect to the postcondition "$y = ax^2 + bx + c$" for the following (assume a, b, c, y, and x are integer variables and that a, b, c, and x are defined):

```
y := ax
y := (y+b)*x
y := y+c
```

(Hint: Be careful regarding x =0.)