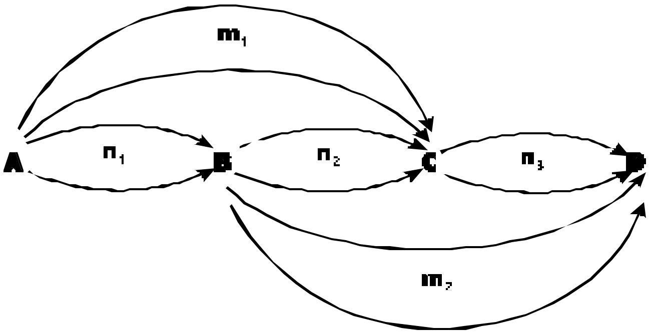
CS 336
Final Examination Solutions

1. a. [5] For $n \ge 1$, how many Boolean (i.e. true- or false-valued) functions exist for n Boolean variables?



The domain of such a function is $\{True, False\}^n$, a set of cardinality 2^n. For a given function, there are two options for the value defined for each variable – thus there are $2^{(2^n)}$ such functions.

2. [10] For $n \ge 1$, how many five-tuples $\langle i, j, k, l, m\rangle$ of non-negative values $i, j, k,$ and l satisfy $i + j + k + l + m \le n$? (Hint: First consider the situation $i + j + k + l + m = n$ and then think about $p = n - (i + j + k + l + m)$.)

Consider placing n indistinguishable balls into six bins labeled i, j, k, l, m and p. Since the number of balls in the p bin is non-negative, each such placement corresponds to a single selection of a five-tuple $\langle i, j, k, l, m\rangle$ of non-negative values i, j, k, l and m satisfying $i + j + k + l + m \le n$. There are $\binom{n+5}{5}$ such placements of n indistinguishable balls into six bins, therefore the same number of five-tuples $\langle i, j, k, l, m\rangle$ of non-negative values i, j, k, l and m satisfying $i + j + k + l + m \le n$.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 1$ and $m \geq 2$:

$$\sum_{k=0}^{n} \binom{n}{k} (m-1)^k = m^n$$

Consider strings of length n selected from the integers $\{1,2,\ldots,m\}$ with repetition allowed. For each of n positions there are m choices, so there are m^n such strings. Alternatively, let k indicate the number of copies of m in the string. The value of k varies from 0 to n. For a fixed value of k there are $\binom{n}{k}$ selections for the placement of the ms and then $(m-1)$ choices for the integers $\{1,2,\ldots,m-1\}$ in each of the $n-k$ remaining positions. Thus there are $\binom{n}{k}(m-1)^{n-k}$ such strings with k copies of m, and $\sum_{k=0}^{n}\binom{n}{k}(m-1)^{n-k}$ overall. This must equal m^n.

b. [10] Using a combinatorial argument, prove that for $n \geq k \geq 0$:

$$\binom{n}{k} k!(n-k)! = n!$$

Consider permutations of length n selected from the integers $\{1,2,\ldots,n\}$. There are $n!$ such permutations. Alternatively, let k satisfy $n \geq k \geq 0$ and for any permutation first select the positions to be occupied by $\{1,2,\ldots,k\}$. There are $\binom{n}{k}$ such selections. Now permute the values $\{1,2,\ldots,k\}$ - there are $k!$ such permutations. Finally, permute the $n-k$ values $\{k+1,k+2,\ldots,n\}$, which can be done in $(n-k)!$ ways, and place them into the positions of the permutation notoccupied by the values from $\{1,2,\ldots,k\}$. Thus, there are $\binom{n}{k}k!(n-k)!$ such permutations and this must equal $n!$.

4. a. [10] For $n \geq 5$, what is the probability that a string of n zeros and ones has exactly 5 ones. (You may assume all strings of n zeros and ones are equally probable.)

b. [5] For $n \geq 5$, what is the probability that a string of n zeros and ones has exactly 5 ones given that it has at least 4 ones. (You may assume all strings of n zeros and ones are equally probable.)

5. **[15]** Prove: If A is a nonempty set, $\mathsf{P}(A)$, the power set of A, is not countably infinite.

Suppose there was a set A such that $\mathsf{P}(A)$ were countably infinite. A could not be finite since then $|\mathsf{P}(A)| = 2^{|A|}$ and so $\mathsf{P}(A)$ would be finite as well. A could not be uncountably infinite since the mapping $f : A \to \mathsf{P}(A)$ defined by $f(a) = \{a\}$ maps A one-to-one into $\mathsf{P}(A)$, so by Theorem 10, $\mathsf{P}(A)$ must be uncountably infinite. Lastly, suppose A is countably infinite. Let $g : \mathbb{N} \xrightarrow[onto]{1-1} A$ and $h : \mathbb{N} \xrightarrow[onto]{1-1} \mathsf{P}(A)$, then g is invertible so $h \circ g^{-1} : A \xrightarrow[onto]{1-1} \mathsf{P}(A)$. Define $\overline{A} = \{a \in A \mid a \notin h \circ g^{-1}(a)\}$. Since $\overline{A} \in \mathsf{P}(A)$, let $\overline{a} = (h \circ g^{-1})^{-1}(\overline{A})$ (that is $\overline{a}$ satisfies $h \circ g^{-1}(\overline{a}) = \overline{A}$). If $\overline{a} \in \overline{A}$ there is a contradiction since then by the definition of $\overline{A}$, $\overline{a} \notin \overline{A}$. Yet if $\overline{a} \notin \overline{A}$ then for the same reason $\overline{a} \in \overline{A}$. Thus either way, there is a contradiction and the assumption that A is countably infinite is false. Since A cannot be finite, uncountably infinite, or countably infinite, A does not exist.

6. a. [10] Prove this corollary to Theorem 6:

Given a countably infinite collection of finite sets $\{A_i\}_{i \in \mathbb{N}}$ *satisfying* $A_0 \neq \varnothing$ *and for* $i \geq 1$,

$$A_i \not\subset \bigcup_{j=0}^{i-1} A_j$$

the union $\bigcup_{i \in \mathbb{N}} A_i$ *is countably infinite. (In other words, if each set contains at least one element not contained in its predecessors, the union cannot be finite.)*

Theorem 6 guarantees that $\bigcup_{i \in \mathbb{N}} A_i$ is countable. For each $i \in \mathbb{N}$, select $a_i \in A_i \sim \bigcup_{j=0}^{i-1} A_j$. Define $f : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} A_i$ by $f(i) = a_i$. For $i_1 \neq i_2$, assume without loss of generality that $i_1 < i_2$, then $f(i_1) = a_{i_1} \in A_{i_2} \subseteq \bigcup_{j=0}^{i_2-1} A_j$ but $f(i_2) = a_{i_2} \notin \bigcup_{j=0}^{i_2-1} A_j$, so $f(i_1) \neq f(i_2)$ and f is one-to-one. By Theorem 4, $\bigcup_{i \in \mathbb{N}} A_i$ is infinite and thus countably infinite.

7. [10] Prove that if $f, g,$ and h are real-valued functions defined on the natural numbers, then $f = o(g)$ and $g = O(h)$ imply $f = o(h)$.

Since $g = O(h)$, there exist non-negative constants M and N_1 such that for all $n \geq N_1$, $|g(n)| \leq M|h(n)|$. Suppose we are given a positive e. Since $f = o(g)$ there exists a non-negative constant N_2 such that for all $n \geq N_2$,

$\left|f(n)\right| \le \frac{e}{M}\left|g(n)\right|$. But then we have for $n \ge \max\{N_1, N_2\}$,

$\left|f(n)\right| \le \frac{e}{M}\left|g(n)\right| \le \frac{e}{M} M \left|h(n)\right| = e\left|h(n)\right|$. We conclude that $f = o(h)$.

8. [10] . Prove that if $0 < a < b$**,** then $n^b \ne \mathrm{O}(n^a)$

Suppose $n^b = \mathrm{O}(n^a)$ and thus there exist non-negative constants M and N such that for all $n \ge N$, $\left|n^b\right| \le M\left|n^a\right|$. We note that since $a < b$, $M^{\frac{1}{b-a}}$ exists and is positive. Choose $n = \max\{N, \left\lceil M^{\frac{1}{b-a}} \right\rceil + 1\}$. We then have $n \ge N$ and $n > M^{\frac{1}{b-a}}$, so $n^{b-a} > M$ and $|n^b| = n^b > Mn^a = M|n^a|$. This is a contradiction so $n^b \ne \mathrm{O}(n^a)$.

9. [10] Assuming x and y are integer variables, prove correct with respect to precondition "y is defined" and postcondition "$x \ne y$":

```
if y > 3 then
    x := y+6
    if x > 11 then
        y : = 11
    endif
else
    x := y-2
    y := y-1
endif
```

```
__________________________ $y$ is defined
if y > 3 then
        ___________________ $y > 3$
        x := y+6
        ___________________ $(y > 3) \wedge (x = y+6)$
        ___________________ $(y > 3) \wedge (x = 9)$
        if x < 11 then
                ____________ $(y > 3) \wedge (x = 9) \wedge (x < 11)$
                ____________ $(y > 3) \wedge (x = 9)$
                y : = 11
                ____________ $(y = 11) \wedge (x = 9)$
                ____________ $x \ne y$
        endif
        ___________________ $(x \ne y) \vee ((y > 3) \wedge (x = 9) \wedge (x \ge 11))$
        ___________________ $(x \ne y) \vee false$
```

```
        __________________ $x \neq y$
else
        __________________ $y \leq 3$
        x := y-2
        __________________ $(y \leq 3) \wedge (x = y - 2)$
        __________________ $x = y - 2$
        y := y-1
        __________________ $(y = y' - 1) \wedge (x = y' - 2)$
        __________________ $x = y - 1$
        __________________ $x \neq y$
endif
__________________________ $(x \neq y) \vee (x \neq y)$
__________________________ $x \neq y$
```

10. [10] Prove the following code is partially correct with respect to precondition "true" and postcondition "$x = 1$" (assume x is an integer variable.):

```
x := 0
while x = 0 do
        x := 1
endwhile
```

Be explicit about your loop invariant: I =

11. a. [10] Prove the following code is partially correct with respect to precondition "$n \geq 1$" and postcondition "$(k/2 < n) \wedge (k \geq n) \wedge (\exists j \geq 0 \ni k = 2^j)$" (assume k and n are integer variables.):

```
k := 1
while k < n  do
          k := 2*k
endwhile
```

Be explicit about your loop invariant: I = $(k/2 < n) \wedge (\exists j \geq 0 \ni k = 2^j)$

```
______________________ $n \geq 1$
k := 1
______________________ $(n \geq 1) \wedge (k = 1)$
______________________ $(k/2 < n) \wedge (\exists j \geq 0 \ni k = 2^j)$
while k < n  do
          _______________ $(k/2 < n) \wedge (\exists j \geq 0 \ni k = 2^j) \wedge (k < n)$
          _______________ $(k < n) \wedge (\exists j \geq 0 \ni k = 2^j)$
          k := 2*k
          _______________ $(k' < n) \wedge (\exists j \geq 0 \ni k' = 2^j) \wedge (k = 2k')$
          _______________ $(k/2 < n) \wedge (\exists j \geq 0 \ni k = 2^j)$
endwhile
______________________ $(k/2 < n) \wedge (k \geq n) \wedge (\exists j \geq 0 \ni k = 2^j)$
```

b. [5] Prove that the loop terminates.

12. [10] Assuming max, a, b, and c are integer variable and that a, b, and c are defined, determine the weakest precondition with respect to the postcondition

“$(\text{min} = a \vee \text{min} = b \vee \text{min} = c) \wedge (\text{min} \le a) \wedge (\text{min} \le b) \wedge (\text{min} \le c)$”:

```
if b < a then
    {if b < c then
        min : = b
    else
        min := c}
else
    {if c < a then
        min : = c}
```

13. a. [10] Determine the weakest precondition with respect to the postcondition "$z = 2$" for the following (assume *z*, *y*, and *x* are integer variables). Simplify your answer so that there are NO logical operators.

```
x := 3
z := 2*x-y
if y>0 then
        z := z-2
else
        z := -z
endif
```

b. [5] Determine the weakest precondition with respect to the postcondition "$(x = y) \wedge (y = x')$" for the following (assume *y*, and *x* are integer variables and are defined):

```
        x = y
```