

Examination 1 Solutions
CS 336

1. [5] Given sets A and B , each of cardinality $n \geq 1$, how many functions map A in a one-to-one fashion onto B ?

Let $A = \{a_1, a_2, \dots, a_n\}$ and $f : A \xrightarrow[\text{onto}]{1-1} B$. There are n options for the value of $f(a_1)$ and, given that, $n-1$ options for the value of $f(a_2)$, ..., and one option for the value of $f(a_n)$. Since B also has cardinality n this function is automatically onto. Thus, there are $n!$ such one-to-one and onto functions.

2 a. [5] Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 1$ exist (allowing repetitions)?

For each of n positions there are r options. Thus, there are r^n such strings.

b. [10] Given the set of r symbols $\{a_1, a_2, \dots, a_r\}$, how many different strings of length $n \geq 2$ exist that contain at least one a_1 and at least one a_2 ? (Assume $r \geq 2$.)

There are $(r-1)^n$ strings that avoid the element a_1 and the same number that avoid a_2 . There are $(r-2)^n$ subsets that avoid both elements so there are $2(r-1)^n - (r-2)^n$ strings that either avoid a_1 or avoid a_2 . Considering the complement, there are $r^n - 2(r-1)^n + (r-2)^n$ strings that contain at least one a_1 and at least one a_2 .

3. [10] Present a combinatorial argument that for all positive integers n :

$$3^n = \sum_{k=0}^n \binom{n}{k} 2^k.$$

Consider as a model strings of length n using the characters from the set $\{a, b, c\}$. For each n positions there are 3 options so there are 3^n such strings. Alternatively, let k represent the number of positions in the string not occupied by a (i.e., thus, occupied by either b or c). The value of k can vary between 0 and n . For a fixed number k of b s and c s, there are $\binom{n}{k}$ ways to determine the positions to be occupied by the b s and c s and then 2 choices (either b or c) for each of these k positions, for a total of $\binom{n}{k} 2^k$ possibilities. The remaining $n-k$ positions must be

occupied by a s. Summing over all possible values of k . We have $\sum_{k=0}^n \binom{n}{k} 2^k$ such strings and this must equal 3^n .

b. [10] Present a combinatorial argument that for all integers $n \geq 3$:

$$\binom{3n}{3} = 3 \binom{n}{3} + 3 \cdot 2n \binom{n}{2} + n^3$$

(Hint: Consider three pairwise disjoint sets of cardinality n .)

Let A, B , and C be pairwise disjoint sets of cardinality n . Consider as a model the number of subsets of $A \cup B \cup C$ of cardinality 3. Since the cardinality of

$A \cup B \cup C$ is $3n$, there are $\binom{3n}{3}$ such subsets of cardinality 3. Now consider that

all three elements could come from the same set A, B , or C , that two could come from one and one comes from another, and that each of the three could come from

a different set. In the first case, there are 3 options for the set and then $\binom{n}{3}$ ways of

selecting the subset. In the second case, there are there are 3 options for the set

from which 2 elements are selected, then $\binom{n}{2}$ ways of selecting those two elements,

and 2 choices for the set from which only one element is selected, and finally n options for that selection. In the final case, there are n possible selections from

each of the three sets. The total is $3 \binom{n}{3} + 3 \cdot 2n \binom{n}{2} + n^3$ and this must equal $\binom{3n}{3}$.

4. [10] A multiset is similar to a set in that order is irrelevant but multiple copies of elements are allowed. For example, the **sets** $\{1, 2, 3\}$ and $\{1, 1, 1, 2, 2, 3\}$ are identical and each has cardinality three but the **multisets** $\{1, 2, 3\}$ and $\{1, 1, 1, 2, 2, 3\}$ are different and the first has cardinality three but the second has cardinality six. How many multisets of cardinality n are there that employ elements from a_1, a_2, \dots, a_r ?

Let us label r bins a_1, a_2, \dots, a_r and consider the number of ways of placing n indistinguishable balls into the r bins. The placements of balls into bins is in one-to-one correspondence with multisets of cardinality n that employ elements from

a_1, a_2, \dots, a_r . There are $\binom{n+r-1}{n}$ such placements of balls in bins so there is the same number of multisets.

5 a. [5] How many strings are there of length $k \geq 1$ using elements from the set $\{1, 2, \dots, n\}$ if repetition is not allowed. (Assume $k \leq n$).

Since there are n options for the first element of the string, $n-1$ options for the second element of the strings, ..., and $n-(k-1)$ options for the last element there are $n \cdot (n-1) \cdots (n-(k-1)) = \frac{n!}{(n-k)!}$ such strings.

b. [10] Now assume each of the different strings in part **a.** is equally likely. What is the probability that the minimum of the k elements is less than or equal to r , where $1 \leq r \leq n-k+1$?

Now we must count how many of these strings have minimum of the k elements is less than or equal to r . Alternatively we could count how many of these strings have minimum of the k elements is strictly greater than r . If the minimum of the k elements is strictly greater than r , then there are only $n-r$ choices for first element of the string, $n-r-1$ options for the second element of the strings, ..., and $n-r-(k-1)$ options for the last element. So there are

$(n-r) \cdot (n-r-1) \cdots (n-r-(k-1)) = \frac{(n-r)!}{(n-k)!}$ such strings. The probability of

such a string is $\frac{(n-r)!}{n!}$ and the probability that the minimum of the k elements is

less than or equal to r is $1 - \frac{(n-r)!}{n!}$.

6. a. [5] a. How many permutations of $a, b, c, d,$ and e have both a to the left of b and b to the left of c ?

There are 5 positions in the string for the d and then 4 positions for the e . Once these are fixed, the positions for $a, b,$ and c are determined since a must be to the left of b and b to the left of c and these must fill the three remaining positions. Thus there are $5 \cdot 4 = 20$ such permutations.

b. [10] Assume all such permutations are equally likely, what is the probability that the permutation begins with a given that it has both a to the left of b and b to the left of c ?

If the permutation begins with a , then there are only 4 positions in the string for the d and then 3 positions for the e . Thus there are 12 such strings with a in the first position and b to the left of c (a will automatically be to the left of b). The probability that the permutation begins with a given that it has both a to the left of b and b to the left of c is $12/20$.