

Examination 2 Solutions

CS 336

1. [20] Using only Definition 2', prove that the set of finitely long strings using characters from $\{A, B, C, \dots, Z, a, b, c, \dots, z\}$ is infinite.

Let $S =$ set of finitely long strings of $\{A, B, C, \dots, Z, a, b, c, \dots, z\}$. For $s \in S$, define $f : S \rightarrow S$ by $f(s) = A|s$ (i.e. the string consisting of A concatenated with s). For $s, t \in S$, if $s \neq t$, then $f(s) = A|s \neq A|t = f(t)$, so f is one-to-one. Let $u = \langle B \rangle$. Since for all $s \in S$ the first element of $f(s)$ is an A , there is no $s \in S$ such that $f(s) = u$. We have then that f maps S into $S \sim \{u\}$, which is a proper subset of S , and by Definition 2', S is infinite.

2. [20] Consider this theorem (that relies upon the Axiom of Choice):

If $f : A \xrightarrow{\text{onto}} B$, then there exists a subset \hat{A} of A such that $f : \hat{A} \xrightarrow{\text{onto}} B$.

Use his theorem to prove: If $f : A \xrightarrow{\text{onto}} B$, and B is infinite then A is infinite.

If $f : A \xrightarrow{\text{onto}} B$, then there exists a subset \hat{A} of A such that $f : \hat{A} \xrightarrow{\text{onto}} B$ thus $f^{-1} : B \xrightarrow{\text{onto}} \hat{A}$. If B is infinite then by Theorem 4 \hat{A} is infinite and by Theorem 3 A is infinite.

3. [20] Is the set of infinitely long strings using characters from $\{A, B, C, \dots, Z, a, b, c, \dots, z\}$ finite, countably infinite, or uncountably infinite? Prove your claim.

The set is uncountably infinite. Let \mathbb{S} denote the set of infinitely long strings using characters from $\{A, B, C, \dots, Z, a, b, c, \dots, z\}$ and \mathbb{B} denote the set of infinitely long bit strings. Define $f : \mathbb{B} \rightarrow \mathbb{S}$ by : $f(s)$ for a bit string s replaces every 0 in the string with the character A and every 1 in the string with the character B . This function is one-to-one since if s and t are bit strings and $s \neq t$, then s and t must differ in some position k , (one having a 0 and one having a 1) but then $f(s)$ and $f(t)$ must differ in that same position k , (one having an A and one having a B). Thus, $f(s) \neq f(t)$ and f is one-to-one. The set \mathbb{B} is uncountably infinite by Theorem and thus, by Theorem , \mathbb{S} is uncountably infinite.

4. [20] Prove that $\sqrt{n^3 + 1} = o(n^2)$.

Given any $\epsilon > 0$, let $N = \max\{1, 2/\epsilon^2\}$. Notice then for $n \geq N$, we have $n \geq 1$

and $n \geq 2/\epsilon^2$. Thus $\frac{2}{n} \leq \epsilon^2$ and $\frac{n^2 + 1}{n^4} \leq \frac{2n^2}{n^4} \leq \epsilon^2$. So $\frac{\sqrt{n^2 + 1}}{n^2} = \sqrt{\frac{n^2 + 1}{n^4}} \leq \epsilon$ and

$|\sqrt{n^3 + 1} - \sqrt{n^3}| \leq \sqrt{n^3 + 1} \leq \epsilon n^2 = \epsilon |n^2|$. Therefore, $\sqrt{n^3 + 1} = o(n^2)$.

5. [20] Employing induction and Theorem 4, prove that for $k \geq 1$, if for $i = 1, 2, \dots, k$, $f_i = O(g)$, then $\prod_{i=1}^k f_i = O(g^k)$.

For $k=1$, we have $\prod_{i=1}^1 f_i = f_1 = O(g) = O(g^1)$ by hypothesis. Now assume $\prod_{i=1}^k f_i = O(g^k)$ and consider $\prod_{i=1}^{k+1} f_i = \prod_{i=1}^k f_i \cdot f_{k+1}$. Since $\prod_{i=1}^k f_i = O(g^k)$ and $f_{k+1} = O(g)$, Theorem 4 guarantees that $\prod_{i=1}^{k+1} f_i = O(g^k \cdot g) = O(g^{k+1})$.

6. [20] Prove that polynomials are asymptotically dominated by their largest power: That is, for $k \geq 0$, $\sum_{i=0}^k a_i n^i = O(n^k)$.

By Theorem 6, $0 \leq i \leq k, n^i = O(n^k)$. By Theorem 1, $0 \leq i \leq k, a_i n^i = O(n^k)$. By Corollary 3.2, $k \geq 0, \sum_{i=0}^k a_i n^i = O(n^k)$.