CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the
comments in boxes

1. a. [5] How many strings of length $n \geq 0$ using characters *a*, *b* or *c* with possible repetition, have exactly n_a *a*s and n_b *b*s (where $n_a + n_b \leq n$)?

Into the n positions of the string, there are $\binom{n}{n_a}$ selections for the positions of the n_a *a*s and, given that, $\binom{n-n_a}{n_b}$ selections for the positions of the n_b *b*s. Once the positions for the *a*s and *b*s are fixed, the positions for the *c*s is determined. Thus there are $\binom{n}{n_a}\binom{n-n_a}{n_b} = \binom{n}{n_a \quad n_b}$ such strings.

b. [10] How many strings of length $n \geq 0$ using characters *a*, *b* or *c* with possible repetition, have either exactly n_a *a*s or exactly n_b *b*s or both (where $n_a + n_b \leq n$)?

For the case of exactly n_a *a*s, into the n positions of the string, there are $\binom{n}{n_a}$ selections for the positions of the n_a *a*s and, given that, 2^{n-n_a} selections for the positions of the *b*s and *c*s. For the case of exactly n_b *b*s, into the n positions of the string, there are $\binom{n}{n_b}$ selections for the positions of the n_b *b*s and, given that, 2^{n-n_b} selections for the positions of the *a*s and *c*s. From above, we have that there are $\binom{n}{n_a}\binom{n-n_a}{n_b}$ strings with exactly n_a *a*s and n_b *b*s, thus there are

$$\binom{n}{n_a}+\binom{n}{n_b}-\binom{n}{n_a}\binom{n-n_a}{n_b}$$

strings with exactly n_a *a*s or exactly n_b *b*s or both.

2. [10] For $n \geq 1$, how many four-tuples $\langle i, j, k, l \rangle$ of non-negative values $i, j, k,$ and l satisfy $i + j + k + l \leq n$? (Hint: First consider the situation $i + j + k + l = n$ and then think about $m = n - (i + j + k + l)$.)

> Consider placing n indistinguishable balls into five bins labeled $i, j, k, l,$ and m. Since the number of balls in the m bin is non-negative, each such placement corresponds to a single selection of a four-tuple $\langle i, j, k, l \rangle$ of non-negative values $i, j, k,$ and l satisfying $i + j + k + l \leq n$. There are $\binom{n+4}{4}$ such placements of n indistinguishable balls into five bins, therefore the same number of four-tuples $\langle i, j, k, l \rangle$ of non-negative values $i, j, k,$ and l satisfying $i + j + k + l \leq n$.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 1$ and $m \geq 2$:

$$\sum_{k=0}^{n} \binom{n}{k} (m-1)^{n-k} = m^n$$

> Consider strings of length n selected from the integers $\{1, 2, \ldots, m\}$ with repetition allowed. For each of n positions there are m choices, so there are m^n such strings. Alternatively, let k indicate the number of copies of m in the string. The value of k varies from 0 to n. For a fixed value of k there are $\binom{n}{k}$ selections for the placement of the m s and then $(m-1)$ choices for the integers $\{1, 2, \ldots, m-1\}$ in each of the $n-k$ remaining positions. Thus there are $\binom{n}{k}(m-1)^{n-k}$ such strings with k copies of m, and $\sum_{k=0}^{n} \binom{n}{k} (m-1)^{n-k}$ overall. This must equal m^n.

b. [10] Using a combinatorial argument, prove that for $n \geq k \geq 0$:

$$\binom{n}{k} k!(n-k)! = n!$$

> Consider permutations of length n selected from the integers $\{1, 2, \ldots, n\}$. There are $n!$ such permutations. Alternatively, let k satisfy $n \geq k \geq 0$ and for any permutation first select the positions to be occupied by $\{1, 2, \ldots, k\}$. There are $\binom{n}{k}$ such selections. Now permute the values $\{1, 2, \ldots, k\}$ - there are $k!$ such permutations. Finally, permute the $n-k$ values $\{k+1, k+2, \ldots, n\}$, which can be done in $(n-k)!$ ways, and place them into the positions of the permutation notoccupied

by the values from $\{1,2,...,k\}$. Thus, there are $\binom{n}{k}k!(n-k)!$ such permutations and this must equal $n!$.

4. a. [10] For $3 \le m \le n$, what is the probability that a string of length m selected without repetition from $\{1,2,...,n\}$ contains the substring $\langle 1,2,3 \rangle$? (You may assume all strings of length m selected without repetition from $\{1,2,...,n\}$ are equally probable.)

There are $\frac{n!}{(n-m)!}$ such equally probable strings. To count the number containing the substring $\langle 1,2,3 \rangle$, consider that we first position the substring $\langle 1,2,3 \rangle$. There are $m-2$ positions for the initial 1, so there are $m-2$ positions for the substring. The remainder of the $m-3$ positions of the string consists of characters from $\{4,5,...,n\}$ of size $n-3$. Thus, there are $(m-2)\frac{(n-3)!}{((n-3)-(m-3))!}$ strings of length m selected without repetition from $\{1,2,...,n\}$ containing the substring $\langle 1,2,3 \rangle$. The probability of such a string is $(m-2)\frac{(n-3)!}{(n-m)!}/\frac{n!}{(n-m)!}$. (This value equals $\frac{m-2}{n(n-1)(n-2)}$ and an alternative argument results in this expression directly.)

5. [10] Using definition 2' (and no cardinality theorems) show that the set of reciprocals of positive integers (i.e., $\{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$) is infinite**.**

Consider the mapping $f:\{1/p \mid p \in \mathbb{Z} \wedge p > 0\} \to \{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$, defined by $f(\frac{1}{p}) = \frac{1}{p+1}$, for $p \in \mathbb{Z}$ and $p > 0$. Since for $\frac{1}{p_1} \ne \frac{1}{p_2}$, $p_1 \ne p_2$ then $p_1 + 1 \ne p_2 + 1$ and $f(p_1) = \frac{1}{p_1} \ne \frac{1}{p_2} = f(p_2)$. The mapping is one-to-one. Lastly $1 = \frac{1}{1} \in \{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$ but if $f(\frac{1}{p}) = \frac{1}{p+1} = 1$ then $p = 0$, but $0 \notin \{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$, so no value exists such that $f(\frac{1}{p}) = 1$ and f maps into a strict subset of $\{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$. Therefore by Definition 2'
$\{1/p \mid p \in \mathbb{Z} \wedge p > 0\}$ is infinite.

6. a. [10] Let $A = \{a,b,c,...,z,A,B,C,...,Z\}$ and and let B be the set of finite strings from A, that is $B = \{\langle \alpha_1, \alpha_2, ... \alpha_n \rangle \mid n \in \mathbb{N} \wedge \alpha_i \in A \; for \; i = 1,2,...,n\}$. Is the set B finite, countably infinite, or uncountably infinite? Prove your claim.

B is countably infinite. For $n \in \mathbb{N}$ define B_n to be the strings from A of length exactly n (i.e. $B_n = \{\langle \alpha_1, \alpha_2, \ldots \alpha_n \rangle | \alpha_i \in A \, for \, i = 1,2,\ldots,n\}$). The cardinality of B_n is 52^n and thus B_n is finite. However $B = \bigcup_{n \in \mathbb{N}} B_n$ thus by Theorem B is countable.

B contains the subset $\{\langle \; \rangle, \langle a \rangle, \langle aa \rangle, \ldots\}$ (i.e. the set of strings of a s of length n for every $n \in \mathbb{N}$). This set is infinite, thus by Theorem B is infinite. We conclude B is countably infinite

b. [5] Prove that the set of finite sets of real values from $[0,1]$
$C = \{\{x_1, x_2, \ldots, x_n\} \,|\, n \in \mathbb{N} \wedge x_i \in [0,1] for\, i = 1,2,\ldots,n\}$ is uncountably infinite.

Consider the mapping $f : [0,1] \to C$ defined by $f(x) = \{x\}$. Since for $x_1 \neq x_2$, $f(x_1) = \{x_1\} \neq \{x_2\} = f(x_2)$. The mapping is one-to-one. By Theorem 11, C is uncountably infinite.

7. [10] Prove that if $f_1 = O(g_1)$ and $f_2 = o(g_2)$, then $f_1 f_2 = o(g_1 g_2)$.

There exist M and N_1 so that for $n \geq N_1, |f_1(n)| \leq M\,|g_1(n)|$. Given $\varepsilon > 0$, there exists N_2 so that for $n \geq N_2, |f_2(n)| \leq \frac{\varepsilon}{M}|g_2(n)|$, thus for

$n \geq \max\{N_1, N_2\}, |f_1(n) f_2(n)| \leq M\,|g_1(n)|\frac{\varepsilon}{M}|g_2(n)| = \varepsilon\,|g_1(n) g_2(n)|$, so $f_1 f_2 = o(g_1 g_2)$.

8. [10] . For a fixed value of k, define $f(n) = \binom{n}{k}$. Prove that $f(n) = \mathrm{O}\,(n^k)$

For $n \geq k$, $|f(n)| = \binom{n}{k} = \frac{1}{k!} n \cdot (n-1) \cdots (n-k+1) \leq \frac{1}{k!} n^k = \frac{1}{k!}|n^k|$, so $f(n) = \mathrm{O}\,(n^k)$.

9. [10] Assuming x and y are integer variables, prove correct with respect to precondition "$x \geq 0$ and y is defined" and postcondition "$x + y \neq 11$":

```
if y > 2 then
        x := y+6
        if x < 10 then
                y : = 1
        endif
else
```

```
        x := y+4
endif
```

10. a. [10] Prove the following code is partially correct with respect to precondition "$n \geq 0$" and postcondition "$s = \sum_{i=1}^{n} a_i b_i$" (assume k and s are integer variables and a and b are integer arrays of length at least n.):

```
        k := 1
        s := 0
        while k ≤ n do
                s := s + (a[k]*b[k])
                k := k+1
        endwhile
```

Be explicit about your loop invariant.

...b. [5] Prove that the loop terminates.

11. a. [10] Determine the weakest precondition with respect to the postcondition "$w > 0$" for the following (assume w, z, y, and x are integer variables and that y and z are defined):

```
x := y
y := x
x := z
y := x
w := x+y+z
```

b. [5] For the same piece of code, determine the weakest precondition with respect to the postcondition "$wy = 12$"

12. [10] Determine the weakest precondition with respect to the postcondition "$y = 1$" for the following code (assume *z*, *y*, and *x* are integer variables and that x and z are defined):

```
if x<3 then
        y := z
        if y<z then
                y := 2*y
        endif
else
        y := z-y
endif
```