3. b. Present a combinatorial argument that for all positive integers n::

$$\binom{2n}{n} = 2\binom{n}{2} + n^2$$

Consider two distinct sets *A* and *B* each of size *n*. Since they are distinct, the cardinality of $A \cup B$ is 2*n*. The number of ways of choosing a pair of elements from $A \cup B$ is $\binom{2n}{2}$. Alternatively, recognize that to get such a pair of elements from $A \cup B$, one might choose both from *A*, both from *B*, or one from each. If both come from *A*, there are $\binom{n}{2}$ possibilities. We get the same number if both elements come from *B*. Finally if one element comes from each of *A* and *B*, then there are n^2 possibilities. The total is $2\binom{n}{2} + n^2$ and this must equal $\binom{2n}{2}$.

4. Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Let A and B be disjoint sets of cardinality n each and $C = A \cup B$. How many subsets of C are there of cardinality *n*. We are selecting elements for such a subset without repletion not with concern for order so there are $\binom{2n}{n}$ such subsets. Alternatively, let k represent the number of elements in such a subset that were selected from A. The value of k may vary from 0 to n. There are $\binom{n}{k}$ such selections of the k elements from A. Now select which k elements from B will **not** be in the subset (the k that remain will thus be **in** the subset). There are $\binom{n}{k}$ of selecting these so $\binom{n}{k}^2$ ways of selecting the subset and $\sum_{k=0}^{n} \binom{n}{k}^2$ ways overall. This must equal $\binom{2n}{n}$.

7. a. Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Let *A* = {*a*, *b*, *c*} and consider all strings of length n using elements of *A*. Since there are three options for each component of the string, there are 3^n such strings. Alternatively, consider first consider the positions of any *c*'s in the string. Let k

represent the number of non-*c*'s (i.e., *a*'s and *b*'s) in the string. Clearly *k* could range from 0 through *n*. For a fixed value of *k*, there are $\binom{n}{k}$ ways to choose the positions for the non-*c*'s . Then for each of the *k* positions, there are two options (i.e., *a* or *b*) for the character in the position. The remaining *n-k* positions must be occupied by *c*'s. Thus there are $\binom{n}{k}2^k$ ways to assign elements to the positions with *k* non-*c*'s. The total is $\sum_{k=0}^{n}\binom{n}{k}2^k$ and this must equal 3^n

b. Present a combinatorial argument that for all nonnegative integers *p*, *s*, and *n* satisfying $p+s \le n$

$$\binom{n}{p}\binom{n-p}{s} = \binom{n}{p+s}\binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

Let a set *A* have *n* elements and consider how many ways there are to select disjoint subsets *B* and *C* of *A* so that *B* has p elements and *C* has *s* elements. First we could select the *p* elements for *B* in $\binom{n}{p}$ ways and then select the *s* elements for *C* from the remaining *n-p* elements of $A \sim B$ in $\binom{n-p}{s}$ ways. Together this yields $\binom{n}{p}\binom{n-p}{s}$ such selections. Alternatively, we could first select the *p+s* elements for $B \cup C$ in $\binom{n}{p+s}$ ways and then select the p elements for *B* from $B \cup C$ in $\binom{p+s}{p}$ ways. There are thus $\binom{n}{p+s}\binom{p+s}{p}$ such selections and this must equal $\binom{n}{p}\binom{n-p}{s}$

8. a. Present a combinatorial argument that for all $n \ge 1$:

$$\sum_{k=1}^{n}\binom{n}{k} = 2^n - 1$$

(**Note**: The summation begins with $k = 1$.)

Consider the cardinality of the set of non-empty subsets of a set A of *n* elements. For each element of A, there are two options: either be present in a subset or not. Thus there are 2^n total subsets but one of these is empty so there are $2^n - 1$ non-empty subsets of A. Alternatively, let *k* indicate the cardinality of the subset. Since we are counting non-empty subsets, *k* ranges from 1 to *n*. For a fixed value of *k*, there are $\binom{n}{k}$ ways of selecting the *k* subset elements from the *n* total elements of A. Adding this to include all possible cases of k, we obtain $\sum_{k=1}^{n}\binom{n}{k}$ and this must equal $2^n - 1$.

b. Present a combinatorial argument that for all integers *k* and *n* satisfying $3 \le k \le n$

$$\binom{n}{k} = \binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$$

(**Hint:** Consider three special elements.)

Consider the number of subsets of size *k* of a set B of cardinality *n*. Since $n \ge 3$, we may select three elements b_1, b_2, b_3 of B and let C = B ~ $\{b_1, b_2, b_3\}$. Thus C has cardinality n-3 and B = C $\cup$ $\{b_1, b_2, b_3\}$. We know there are $\binom{n}{k}$ such subsets. Alternatively, to select *k* elements of B for a subset there are four options: all k come from C, *k*-1 come from C and the *k*th is either $b_1, b_2,$ or b_3, *k*-2come from C and the k-1st and *k*th are exactly two of $b_1, b_2,$ or b_3, or *k*-3 come from C and all of $b_1, b_2,$ and b_3 are present. For the first option, there are $\binom{n-3}{k}$ possibilities since all *k* come from C. For the second option, there are $3\binom{n-3}{k-1}$ possibilities, since *k*-1 elements are selected from C and one from the three of $b_1, b_2,$ or b_3. For the third option, there are $3\binom{n-3}{k-2}$ possibilities, since *k*-2 elements are selected from C and one from the three of $b_1, b_2,$ or b_3 is **not** selected. Lastly, if *k*-3 come from C and all of $b_1, b_2,$ and b_3 are present, then there are $\binom{n-3}{k-3}$ options. The total is $\binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$ and this must equal $\binom{n}{k}$

9. Present a combinatorial argument that for all positive integers $m, n,$ and r, satisfying $r \le \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

(**Hint**: Consider selecting from two sets.)

Let A and B be disjoint sets of cardinalities m and n, respectively. Let $C = A \cup B$ and consider the number of subsets of C of cardinality r. Since $|C| = |A| + |B) = m + n$, there are $\binom{m+n}{r}$ such subsets. Alternatively let k be the number of elements in a subset that came from A. The value of k can range from

0 to r. For a fixed value of k, there are $\binom{m}{k}$ ways to select the k elements from A and $\binom{n}{r-k}$ ways to select the remaining $r-k$ elements from B, thus $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ total ways. This must equal $\binom{m+n}{r}$.