Gram-Schmidt Orthonormalization

Given an finite dimensional inner product space G with basis $< g_1, \ldots, g_n >$ determine an orthonormal basis $< \tilde{g}_1, \ldots, \tilde{g}_n >$ for G.

A. "Classical" Algorithm

$$\tilde{g}_1 = \frac{1}{\| g_1 \|} g_1$$

for $i = 2, \ldots, n$

$$\hat{g}_i = g_i - \sum_{j=1}^{i-1} (g_i, \tilde{g}_j) \tilde{g}_j$$

$$\tilde{g}_i = \frac{1}{\| \hat{g}_i \|} \hat{g}_i$$

B. "Modified" Algorithm

$$\tilde{g}_1 = \frac{1}{\| g_1 \|} g_1$$

for $i = 2, \ldots, n$

$\hat{g}_i = g_i$

for $j = 1, \ldots, i-1$

$$\hat{g}_i \leftarrow \hat{g}_i - (\hat{g}_i, \tilde{g}_j) \tilde{g}_j$$

$$\tilde{g}_i = \frac{1}{\| \hat{g}_i \|} \hat{g}_i$$

To solve the least squares problem:

Determine $g^* \in G$ that minimizes $\| f = g \|$ over all $g \in G$

We have $g^* = \sum_{i=1}^{n} (f, \tilde{g}_i) \tilde{g}_i$. However the calculation of the coefficients $a_i = (f, \tilde{g}_i)$ that is "consistent" with the modified algorithm is this:

$a_1 = (f, \tilde{g}_1)$

for $i = 2, \ldots, n$

$$a_i = (f - \sum_{j=1}^{i-1} a_j \tilde{g}_j, \tilde{g}_i)$$

Algebraically the two are equivalent. Computationally, the second is preferred.