Verification of Gram-Schmidt Orthonormalization

Theorem 1 - Gram-Schmidt Classical Algorithm:
Given an finite dimensional inner product space G with basis $< g_1, \ldots, g_n >$, define the set $\{\tilde{g}_1, \ldots, \tilde{g}_n\}$ as follows:

$$\tilde{g}_1 = \frac{1}{\| g_1 \|} g_1$$

for $i = 2, \ldots, n$

$$\hat{g}_i = g_i - \sum_{j=1}^{i-1} (g_i, \tilde{g}_j) \tilde{g}_j$$

$$\tilde{g}_i = \frac{1}{\| \hat{g}_i \|} \hat{g}_i ,$$

then for no i is $\hat{g}_i = 0$, so the classical algorithm proceeds to conclusion with the result that $< \tilde{g}_1, \ldots, \tilde{g}_n >$ is an orthonormal basis for G.

Proof:
We proceed by induction. Clearly $g_1 \neq 0,$ so $\tilde{g}_1$ is defined and

$\|\tilde{g}_1\| = \left\| \frac{1}{\| g_1 \|} g_1 \right\| = \frac{1}{\| g_1 \|} \|g_1\| = 1.$ Now assume for $2 \leq i \leq n$ and $1 \leq j,k \leq i-1$ that

$$(\tilde{g}_j, \tilde{g}_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}.$$

Consider $\hat{g}_i$. We have

$$\begin{aligned}(\hat{g}_i, \tilde{g}_k) &= (g_i - \sum_{j=1}^{i-1} (g_i, \tilde{g}_j) \tilde{g}_j, \tilde{g}_k) \\ &= (g_i, \tilde{g}_k) - \sum_{j=1}^{i-1} (g_i, \tilde{g}_j)(\tilde{g}_j, \tilde{g}_k) \\ &= (g_i, \tilde{g}_k) - (g_i, \tilde{g}_k)(\tilde{g}_k, \tilde{g}_k) \\ &= (g_i, \tilde{g}_k) - (g_i, \tilde{g}_k) \\ &= 0.\end{aligned}$$

Also $\hat{g}_i \neq 0$ since the alternative implies that the set $\{\tilde{g}_1, \ldots, \tilde{g}_{i-1}, \hat{g}_i\}$ is linearly dependent (and thus, that the set $\{g_1, \ldots, g_{i-1}, g_i\}$ is linearly dependent). So

$$\|\tilde{g}_i\| = \left\| \frac{1}{\| \hat{g}_i \|} \hat{g}_i \right\| = \frac{1}{\| \hat{g}_i \|} \|\hat{g}_i\| = 1$$

and

$$
\begin{aligned}
(\tilde{g}_i, \tilde{g}_k) &= (\frac{1}{\| \hat{g}_i \|} \hat{g}_i, \tilde{g}_k) \\
&= \frac{1}{\| \hat{g}_i \|} (\hat{g}_i, \tilde{g}_k) \\
&= 0.
\end{aligned}
$$

We then have that for $1 \le j, k \le i$ that

$$
(\tilde{g}_j, \tilde{g}_k) = \begin{cases} 1 & j = k \\ 0 & j \ne k \end{cases}.
$$

By induction this holds for all his holds for all $1 \le j, k \le n.$ Since the $\{\tilde{g}_1, \ldots, \tilde{g}_n\}$ form a set of n linearly independent elements of G, $< \tilde{g}_1, \ldots, \tilde{g}_n >$ is an basis for G. ?

Theorem 2 - Gram-Schmidt Modified Algorithm:

Given an finite dimensional inner product space G with basis $< g_1, \ldots, g_n >$, define the set $\{\tilde{g}_1, \ldots, \tilde{g}_n\}$ as follows:

$$
\begin{aligned}
&\tilde{g}_1 = \frac{1}{\| g_1 \|} g_1 \\
&\text{for } i = 2, \ldots, n \\
&\qquad \hat{g}_i^1 = g_i \\
&\qquad \text{for } j = 1, \ldots, i-1 \\
&\qquad\qquad \hat{g}_i^{j+1} = \hat{g}_i^j - (\hat{g}_i^j, \tilde{g}_j)\tilde{g}_j \\
&\qquad \tilde{g}_i = \frac{1}{\| \hat{g}_i^i \|} \hat{g}_i^i
\end{aligned}
$$

then for no i is $\hat{g}_i = 0$, so the modified algorithm proceeds to conclusion with the result that $< \tilde{g}_1, \ldots, \tilde{g}_n >$ is an orthonormal basis for G.

Proof:

We will show that the elements $\tilde{g}_1, \ldots, \tilde{g}_n$ computed by this algorithm are identical to those produced by the classical algorithm. Clearly the quantity $\tilde{g}_1$ is identical. For $2 \le i \le n$ assume that the algorithms produce identical elements $\tilde{g}_1, \ldots, \tilde{g}_{i-1}$. We proceed by induction.

Clearly $g_1 \ne 0,$ so $\tilde{g}_1$ is defined and $\|\tilde{g}_1\| = \left\| \frac{1}{\| g_1 \|} g_1 \right\| = \frac{1}{\| g_1 \|} \|g_1\| = 1.$ Notice for $i = 2, \ldots, n$ that $\hat{g}_i^1 = g_i$. Now assume for $i \ge 2$ and $1 \le j \le i-1$ that

$$
\hat{g}_i^j = g_i - \sum_{k=1}^{j-1} (g_i, \tilde{g}_k)\tilde{g}_k \quad .
$$

Since for $1 \le k \le j-1$ $(\tilde{g}_k, \tilde{g}_j) = 0$, we then have

$$\begin{aligned}
\hat{g}_i^{j+1} &= \hat{g}_i^j - (\hat{g}_i^j, \tilde{g}_j)\tilde{g}_j \\
&= g_i - \sum_{k=1}^{j-1}(g_i, \tilde{g}_k)\tilde{g}_k - (g_i - \sum_{k=1}^{j-1}(g_i, \tilde{g}_k)\,\tilde{g}_k, \tilde{g}_j)\tilde{g}_j \\
&= g_i - \sum_{k=1}^{j-1}(g_i, \tilde{g}_k)\tilde{g}_k - (g_i, \tilde{g}_j)\tilde{g}_j + \sum_{k=1}^{j-1}(g_i, \tilde{g}_k)(\tilde{g}_k, \tilde{g}_j)\tilde{g}_j \\
&= g_i - \sum_{k=1}^{j}(g_i, \tilde{g}_k)\tilde{g}_k + \sum_{k=1}^{j-1}(g_i, \tilde{g}_k)(\tilde{g}_k, \tilde{g}_j)\tilde{g}_j \\
&= g_i - \sum_{k=1}^{j}(g_i, \tilde{g}_k)\tilde{g}_k.
\end{aligned}$$

We conclude that for $j = i$, $\hat{g}_i^i = g_i - \sum_{k=1}^{i-1}(g_i, \tilde{g}_k)\,\tilde{g}_k$, but this is the same as $\hat{g}_i$ in the classical algorithm. It follows that the quantity $\tilde{g}_i$ is identical in the two algorithm and, by induction, the entire sequence $\tilde{g}_1, \ldots, \tilde{g}_n$ is identical. Since the same elements are computed, the claims of independence and orthonormality for the elements of the classical algorithm hold for those computed by the modified algorithm. ?