Example of Peano Analysis for Trapezoidal Rule

For $a < b$, we consider the linear functional

$$E(f) = \int_a^b f(z)dz - \frac{b-a}{2}(f(a) + f(b)).$$

The support of E is the interval $[a,b]$. We have $E(1) = 0$ and $E(x) = 0$ but $E(x^2) = -\frac{(b-a)^3}{6}$, so E is exact for polynomials of degree one but not for polynomials of degree two. We compute

$$\begin{aligned} K(t) &= E_x((x-t)_+) \\ &= \int_a^b (x-t)_+ dx - \frac{b-a}{2}((a-t)_+ + (b-t)_+) \\ &= \int_t^b (x-t)dx - \frac{b-a}{2}(0 + (b-t)) \\ &= \frac{(b-t)^2}{2} - \frac{(b-a)(b-t)}{2} \\ &= -\frac{(b-t)(t-a)}{2}, \end{aligned}$$

which is negative since both $b-t$ and $t-a$ are positive on $[a,b]$. From Corollary 2 we have

$$\begin{aligned} E(f) &= \frac{1}{2!}E(x^2)f''(\xi) \\ &= -\frac{(b-a)^3}{12}f''(\xi), \end{aligned}$$

for some $\xi \in [a,b]$.

Extension to Composite Trapezoidal Rule

For $\alpha < \beta$, we consider the linear functional based upon the composite trapezoidal rule:

$$\overline{E}(f) = \int_\alpha^\beta f(z)dz - h(\frac{1}{2}f(\alpha) + f(\alpha+h) + \ldots + f(\beta-h) + \frac{1}{2}f(\beta)),$$

where $h = \frac{\beta-\alpha}{n}$ for some positive n. We could express $\overline{E}$ in terms of the linear functional in the previous analysis by defining for $i = 0,1,\ldots,n-1$

$$E_i(f) = \int_{\alpha+ih}^{\alpha+(i+1)h} f(z)dz - \frac{h}{2}(f(\alpha+ih) + f(\alpha+(i+1)h))$$

and then noting that

$$\overline{E}(f) = \sum_{i=0}^{n-1} E_i(f).$$

According to the conclusion reached above

$$E_i(f) = -\frac{h^3}{12} f''(\xi_i),$$

for $\xi_i \in [\alpha + ih, \alpha + (i+1)h]$ and $i = 0, 1, \ldots, n-1$, thus

$$\begin{aligned}\bar{E}(f) &= -\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) \\ &= -\frac{(\beta-\alpha)h^2}{12n}\sum_{i=0}^{n-1} f''(\xi_i).\end{aligned}$$

However, since f'' is continuous on $[\alpha, \beta]$ and $\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)$ is a value between $\min_{t\in[\alpha,\beta]} f''(t)$ and $\max_{t\in[\alpha,\beta]} f''(t)$, then there exists some $\bar{\xi} \in [\alpha, \beta]$ such that $f''(\bar{\xi}) = \frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)$, so we conclude that

$$\bar{E}(f) = -\frac{(\beta-\alpha)h^2}{12} f''(\bar{\xi}).$$

Higher Order Terms

We have just determined the first power – namely two – in the error expansion for the composite trapezoidal rule. To see how Peano Theory can be used to obtain higher powers, we construct a new functional based upon our analysis of E. Noticing the unknown value $f''(\xi)$ we hypothesize that this might be close to $\frac{f'(b)-f'(a)}{b-a}$ and thus propose the new functional

$$\hat{E}(f) = \int_a^b f(z)dz - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{12}(f'(b)-f'(a)),$$

that is

$$\hat{E}(f) = E(f) + \frac{(b-a)^2}{12}(f'(b)-f'(a)).$$

As before, the support of $\hat{E}$ is the interval $[a,b]$. We have $\hat{E}(1)=0, \hat{E}(x)=0, \hat{E}(x^2)=0,$ and $\hat{E}(x^3)=0,$ but $E(x^4) = \frac{(b-a)^5}{30}$, so $\hat{E}$ is exact for polynomials of degree three but not for polynomials of degree four. Since the derivatives of $(x-t)_+^3$ evaluated at $x=a$ and b are $3(a-t)_+^2 = 0$ and $3(b-t)_+^2 = 3(b-t)^2$, respectively, we compute

$$
\begin{aligned}
\hat{K}(t) &= \hat{E}_x((x-t)_+^3) \\
&= \int_a^b (x-t)_+^3\,dx - \frac{b-a}{2}((a-t)_+^3 + (b-t)_+^3) + \frac{(b-a)^3}{4}(b-t)^2 \\
&= \int_t^b (x-t)^3\,dx - \frac{b-a}{2}(0 + (b-t)^3) + \frac{(b-a)^3}{4}(b-t)^2 \\
&= \frac{(b-t)^4}{4} - \frac{(b-a)(b-t)^3}{2} + \frac{(b-a)^3(b-t)^2}{4} \\
&= \frac{(b-t)^2(t-a)^2}{4}
\end{aligned}
$$

It's clear that this is positive on $[a,b]$. From Corollary 2 we have

$$
\begin{aligned}
\hat{E}(f) &= \frac{1}{4!}\hat{E}(x^4)f^{(4)}(\hat{\xi}) \\
&= \frac{(b-a)^5}{720}f^{(4)}(\hat{\xi}),
\end{aligned}
$$

for some $\hat{\xi} \in [a,b]$.

Now we return to the composite trapezoidal rule. Analogously to above, define

$$
\hat{E}_i(f) = \int_{\alpha+ih}^{\alpha+(i+1)h} f(z)dz - \frac{h}{2}(f(\alpha+ih) + f(\alpha+(i+1)h)) + \frac{h^2}{12}(f'(\alpha+(i+1)h) - f'(\alpha+ih))
$$

and

$$
\hat{\bar{E}}(f) = \sum_{i=0}^{n-1}\hat{E}_i(f).
$$

Noting that because of cancellation in the first derivative terms we have

$$
\begin{aligned}
\hat{\bar{E}}(f) &= \sum_{i=0}^{n-1}E_i(f) + \frac{h^2}{12}(f'(\beta) - f'(\alpha)) \\
&= \bar{E}(f) + \frac{h^2}{12}(f'(\beta) - f'(\alpha)).
\end{aligned}
$$

According to the conclusion reached above

$$
\hat{E}_i(f) = \frac{h^5}{720}f^{(4)}(\hat{\xi}_i),
$$

for $\hat{\xi}_i \in [\alpha+ih, \alpha+(i+1)h]$ and $i = 0,1,\dots,n-1$, thus

$$
\begin{aligned}
\hat{\bar{E}}(f) &= \frac{h^3}{720}\sum_{i=0}^{n-1}f^{(4)}(\hat{\xi}_i) \\
&= \frac{(\beta-\alpha)h^4}{720n}\sum_{i=0}^{n-1}f^{(4)}(\hat{\xi}_i).
\end{aligned}
$$

However, since $f^{(4)}$ is continuous on $[\alpha,\beta]$ and $\frac{1}{n}\sum_{i=0}^{n-1} f^{(4)}(\hat{\xi}_i)$ is a value between $\min_{t\in[\alpha,\beta]} f^{(4)}(t)$ and $\max_{t\in[\alpha,\beta]} f^{(4)}(t)$, then there exists some $\hat{\hat{\xi}} \in [\alpha,\beta]$ such that $f^{(4)}(\hat{\hat{\xi}}) = \frac{1}{n}\sum_{i=0}^{n-1} f^{(4)}(\hat{\xi}_i)$, so we conclude that

$$\hat{\hat{E}}(f) = \frac{(\beta-\alpha)h^4}{720} f^{(4)}(\hat{\hat{\xi}}).$$

But recalling that

$$\hat{\hat{E}}(f) = \bar{E}(f) + \frac{h^2}{12}(f'(\beta) - f'(\alpha)),$$

we have

$$\begin{aligned}\bar{E}(f) &= \hat{\hat{E}}(f) - \frac{h^2}{12}(f'(\beta) - f'(\alpha)) \\ &= \frac{(\beta-\alpha)h^4}{720} f^{(4)}(\hat{\hat{\xi}}) - \frac{h^2}{12}(f'(\beta) - f'(\alpha))\end{aligned}$$

We have now determined the second power – namely four – in the error expansion for the composite trapezoidal rule. The process could be continued to show that

$$\bar{E}(f) = c_1 h^2 + c_2 h^4 + \ldots + \mathrm{O}(h^{2p})$$

For appropriately defined $c_1, c_2,$ etc..