Best Approximation on Normed Linear Spaces

Theorem: Let *S* be a normed linear space with norm $\|.\|$, *G* be a finite dimensional subspace of *S*, and *f* an element of *S*. There exists an element g^* of *G* that minimizes $\|f-g\|$ over all elements $g \in G$.

Proof: Let $\langle g_1, g_2, \ldots, g_n \rangle$ be a basis for *G* and define $N(a_1, a_2, \ldots, a_n) = \left\| f - \sum_{i=1}^{n} a_i g_i \right\|$.

We will show that *N* is a continuous function on R^n, that its infimum over R^n is assumed on a smaller set, and that this smaller set is closed and bounded. That will guarantee that the minimum of *N* is assumed. The coefficients $(a_1^*, a_2^*, \ldots, a_n^*)$ at this minimum are those of the element g^* of *G* that minimizes $\|f-g\|$; that is $g^* = \sum_{i=1}^{n} a_i^* g_i$.

To see that *N* is continuous, notice that

$$
\begin{aligned}
|N(a_1, a_2, \ldots, a_n) - N(b_1, b_2, \ldots, b_n)| &= \left| \left\| f - \sum_{i=1}^{n} a_i g_i \right\| - \left\| f - \sum_{i=1}^{n} b_i g_i \right\| \right| \\
&\le \left\| (f - \sum_{i=1}^{n} a_i g_i) - (f - \sum_{i=1}^{n} b_i g_i) \right\| = \left\| \sum_{i=1}^{n} (a_i - b_i) g_i \right\| \\
&\le (\sum_{i=1}^{n} \|g_i\|) \ \max\{|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|\} .
\end{aligned}
$$

The function $M(a_1, a_2, \ldots, a_n) = \left\| \sum_{i=1}^{n} a_i g_i \right\|$ is obviously also continuous and non-negative. Consider its minimum $\underline{M}$ on the compact set

$$\{(a_1, a_2, \ldots, a_n) | \max\{|a_1|, |a_2|, \ldots, |a_n|\} = 1\} .$$

If $\underline{M} = 0$, then there is a non-zero linear combination of the basis elements $(g_1, g_2, \ldots, g_n)$ that is zero. Since this contradicts their linear independence, this is a contradiction and $\underline{M} > 0$. By linearity, it is easy now to show that

$$\left\| \sum_{i=1}^{n} a_i g_i \right\| \ge \underline{M} \max\{|a_1|, |a_2|, \ldots, |a_n|\}$$

for all sets of coefficients $(a_1, a_2, \ldots, a_n)$. We may conclude that the set

$$A = \{(a_1, a_2, \ldots, a_n) \mid \left\| \sum_{i=1}^{n} a_i g_i \right\| \le 2\|f\|\}$$

is bounded. It is obviously closed and thus compact. Since continuous functions assume their minima on compact sets, the minimum of *N* over *A* is assumed at some set of coefficients $(a_1^*, a_2^*, \ldots, a_n^*)$.

Suppose a set of coefficients $(a_1, a_2, \ldots, a_n) \notin A$. Then $\left\| \sum_{i=1}^{n} a_i g_i \right\| > 2\|f\|$. But then also

$$N(a_1, a_2, \ldots, a_n) = \left\| f - \sum_{i=1}^{n} a_i g_i \right\| \geq \left\| \sum_{i=1}^{n} a_i g_i \right\| - \|f\| > 2\|f\| - \|f\| = \|f\| = N(0,0,\ldots,0).$$

Since the set of coefficients $(0,0,\ldots,0) \in A$, we have

$$N(a_1^*, a_2^*, \ldots, a_n^*) \leq N(0,0,\ldots,0) < N(a_1, a_2, \ldots, a_n)$$

for all $(a_1, a_2, \ldots, a_n) \notin A$. We conclude that the minimum of N over all R^n is assumed at $(a_1^*, a_2^*, \ldots, a_n^*)$ and that $g^* = \sum_{i=1}^{n} a_i^* g_i$ minimizes $\|f - g\|$ over all elements $g \in G$.