CS388C: Combinatorics and Graph Theory — Problem Set No. 1
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1. Fan-in $\ell$ circuits are the same as ordinary circuits except that all gates have $\ell$ inputs. Each gate may be chosen to compute any of the $2^{2^\ell}$ functions on $\ell$ bits (in particular, the gate may ignore some inputs and hence compute a function on fewer bits). Fix a function $\ell = \ell(n) \geq 2\log_2 n$. Call a function on $n$ bits hard if no fan-in $\ell$ circuit of size $2^{n-\ell-2}$ computes it. Show that the fraction of functions on $n$ bits which are hard is $1 - O(2^{-n})$.

2. How many ways can the positive integer $n$ be expressed as an *ordered* sum of (any number of) positive integers? For example, here are four of the several ways to express 4: 4, 3+1, 1+3, and 2+1+1.

3. How many monic polynomials of degree $n$ over the field $\mathbb{F}_p$, $p$ prime, never take the value 0?

4. How many subsets of $[n]$ don't contain two successive integers?

5. For $G$ a graph, let $p_G$ denote the chromatic polynomial, i.e., $p_G(k)$ is the number of proper $k$-colorings of $G$. Show that $p_G(x) = \sum_c a_c x^c$, where the coefficients $a_c$ are expressions involving the quantities $h(e,c)$. Here $h(e,c)$ denotes the number of spanning subgraphs with $e$ edges and $c$ connected components. (Note that a spanning subgraph could contain isolated vertices, each of which is a connected component.)