On the Relationship of Tensors and Products of Categories

Due Monday March 31th, 8pm

We discussed the idea of multi-dimensional models in class. Along each dimension of a n-cube are the features of a particular model. Each dimension represents an orthogonal feature model. Consider the 2-cube (matrix) below. The horizontal dimensional model has {Θ,**X**,**Y**,**Z**}, where Θ is the base program and **X**, **Y**, and **Z** are optional abstract features. The vertical dimension has {Θ,**A**,**B**} where again Θ is the base program and **A** and **B** are optional abstract features. ("Abstract" features mean that their implementation, as a composition of matrix elements, are determined by abstract features selected in another model. For example, the base program is implemented by θ, **a.**θ, or **a.b.**θ, depending on which features in the vertical dimension are selected). Abstract features are denoted in **UPPERCASE**, and primitive elements are in **lowercase**..

Z	Y	X	Θ	
z	y	x	θ	Θ
u	s	q	a	A
v	t	r	b	B

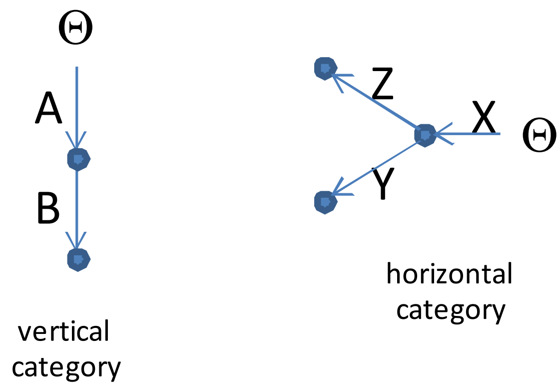
Projections and contractions of this matrix to a scalar yields programs of this product-line. Assume that the dimensions (and "diagonals") are orthogonal, i.e., **x.a=a.x**, and **q.b=b.q**, etc. That is, the refinements that **x** makes to Θ are orthogonal to the refinements that **a** makes to Θ. Contracting the left-most two columns yields:

Z	Y	X·Θ	
z	y	x·θ	Θ
u	s	q·a	A
v	t	r·b	B

and contracting the top-most two rows of the original matrix yields:

Z	Y	X	Θ	
u·z	s·y	q·x	a·θ	A·Θ
v	t	r	b	B

What the above matrices (tensors) do not capture are the compositional relationships between horizontal features, and the compositional relationships between vertical features. We can depict such relationships as a product-line or category (shown below):



That is, **A** can be composed with Θ, and **B** can be composed after **A**. **X** can be composed with Θ, and **Z** and **Y** can be composed after **X**. What this means is that not all combinations of vertical features are permitted, and not all combinations of horizontal features are permitted.

> Hint: assume that each element in the original matrix is a "small" arrow. When you take the product of the horizontal and vertical categories, you will produce a 3-dimensional commuting diagram, where each arrow is a "long" arrow. What you should show is that each long arrow is actually a composition of 1 or more small arrows, and that there is a definite, regular pattern of how long arrows are composed from short arrows.

(1) Explain how the above matrix is a compact representation of a product of the vertical and horizontal categories. Form their product category, and label each long arrow with a short arrow (or composition of arrows) from the matrix. Hint: in case you want to verify your labeling, you might show that equivalent paths using long arrows can be translated into equivalent paths using short arrows. It can become too confusing to recognize commuting relationships among short arrows.

(2) Assuming the orthogonality of dimensions, give a reason why the order in which the dimensions of a matrix (or more generally a tensor) are contracted should not matter.

(3) The original ideas of matrices and tensors was published under the title of "origami". The idea is that a matrix could be reduced (contracted) by composing rows and columns in any order — compose two rows, then compose two columns, then compose another two rows, etc. Although no proof was presented of this idea, is it consistent with your answer in (2)?