Lectures in Discrete Differential Geometry 1 – Plane Curves

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1 What is Discrete Differential Geometry?

The classic theory of differential geometry concerns itself with smooth curves and surfaces. In practice, however, our experiments can only measure a finite amount of data, and our simulations can only resolve a finite amount of detail. Discrete differential geometry (DDG) studies *discrete* counterparts of classical differential geometry that are applicable in this discrete setting, and converge to the smooth theory in the limit of refinement.

Smooth geometric objects possess a rich set of symmetries, invariants, and interrelationships – for instance, the Gauss-Bonnet theorem ties together the Gaussian curvature of a surface to its topology in a beautiful way. There are many different ways to discretize any given geometric object, and surprisingly, it is often possible with care to choose a discretization with special properties that *exactly*, not just approximately, mimic this smooth structure. Choosing the right discretization that preserves the right structure leads to particularly elegant and efficient algorithms for solving problems in computational geometry and physical simulation.

I will give an overview of DDG, with a particular focus on discretizing the geometry of surfaces in $\mathbb{R}^3$. Topics we will cover include discrete curvature measures, discrete exterior calculus, the Laplace-Beltrami operator and its properties, mean curvature flow, conformal surface parameterization, vibration modes of membranes, statics and dynamics of elastic plates, and time integration using the discrete Hamilton's principle.

Before studying discrete surfaces, however, we will look at the geometry of curves in the plane, and in this more elementary setting gain initial experience with DDG.

2 Smooth Curve Geometry

Let $\gamma(t) : [0,1] \to \mathbb{R}^2$ be a curve in the plane. To avoid technical distractions we will assume that γ is C^∞ smooth and regular: $\|\gamma'(t)\| \neq 0$. Most results in this chapter carry over with minor modification to piecewise-regular C^2 curves. We will say the curve is *closed* if the beginning and end agree to every order:

$$\gamma^{(c)}(0) = \gamma^{(c)}(1) \quad \forall c \in \mathbb{Z} \geq 0.$$

We are interested in the *geometry* of γ: the properties of the oriented set of points $\operatorname{im}\gamma$. We distinguish the beginning $\gamma(0)$ and end $\gamma(1)$ of the curve, but otherwise care only about the image of γ, and not about the properties of the function γ itself.

So what about the geometry of γ can we measure? We can locate individual positions $\gamma(t)$, of course, and measure extrinsic distances $\|\gamma(t_2) - \gamma(t_1)\|$ between points. We also have a notion of *intrinsic* distance, the arc length between two points on the curve

$$\int_{t_1}^{t_2} \|\gamma'(t)\| dt.$$

Recall that we can reparameterize curves by arc length. To simplify calculations we will assume from now on that we have done so, so that $\gamma : [0, L] \to \mathbb{R}^2$ has unit-length derivative: $\|\gamma'(s)\| = 1$.

What else can we measure? At every point on the curve we have a tangent vector $T(s) = \gamma'(s)$ and normal vector $N(s) = JT(s)$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acts on vectors in the plane by rotating them counterclockwise by ninety degrees. (We arbitrary choose the orientation of N so that a circle, winding clockwise, has outward-pointing normal vector.) Most importantly, we can measure *curvature* $\kappa(s)$. Often curvature is introduced as the nonnegative magnitude of the derivative of the tangent vector $|\kappa(s)| = \|T'(s)\|$ but we will find it more useful to work with *signed curvature*

$$\kappa(s) = -T'(s) \cdot N(s).$$

Here the sign of $\kappa(s)$ is chosen arbitrary so that a circle, oriented clockwise, has positive curvature.

Remark Knowing the curvature of a curve is enough to reconstruct the entire curve, up to the position $\gamma(0)$ and orientation $\gamma'(0)$ of its starting point. To see this, notice that at every s we can decompose $T'(s)$ in the tangent and normal basis:

$$T'(s) = \alpha T(s) + \beta N(s) = \alpha T(s) + \beta JT(s).$$

Since $\|T\| = 1$, differentiating gives $2T \cdot T' = 0$ and $\alpha = 0$. The definition above of curvature gives $\beta = -\kappa(s)$. Therefore we can integrate $\gamma(s)$ from the system of first-order ODEs

$$\begin{aligned} T'(s) &= -\kappa(s)JT(s) \\ \gamma'(s) &= T(s). \end{aligned}$$

2.1 Properties of Curvature

Let's try to make a list of all of the properties we can think of that curvature satisfies. This list should include some properties about how curvature changes as the curve γ changes; to facilitate this we will think of $L(\gamma)$ as a *functional* which evaluates to the total arc length of γ, and κ_γ as a functional that takes in a curve γ and produces the curvature function $\kappa_\gamma(s) : [0, L(\gamma)] \to \mathbb{R}$ for that curve.

So how about the properties?

0. The identity from the remark above: $\gamma''(s) = -\kappa(s)N(s)$

We will call this "property zero" as it is directly equivalent to the definition of curvature. Next, curvature possesses certain symmetry properties. It is invariant under applying Euclidean motions (translation and rotation) σ to γ:

1. $\kappa_{\sigma \circ \gamma}(s) = \kappa_\gamma, \sigma \in E^2$.

Reflecting γ over a line in the plane, or reparameterizing γ to reverse the direction of travel, flip the sign of curvature. Lastly, curvature scales inversely with length.

2. $\kappa_{r \circ \gamma} = -\kappa_\gamma$ for plane reflections r.
3. $\kappa_{\gamma(L-s)}(s) = -\kappa_\gamma(L - s)$.
4. $\kappa_{\alpha\gamma} = \frac{1}{\alpha}\kappa_\gamma$ for $\alpha \in \mathbb{R}$.

We know the curvature of several special classes of curves:

5. For a straight line, $\kappa = 0$.
6. For a circle of radius r oriented clockwise, curvature is constant and equal to $\frac{1}{r}$.

7. $\kappa(s)$ diverges at "kinks": points where $T(s)$ is undefined or not differentiable. Notice that such kinks cannot occur in a regular curve, but can occur as a limit of regular curves, and curvature also diverges in this limit.

Two more miscellaneous properties. The first is easy to overlook and follows directly from the definition, but is tremendously important:

8. Curvature is *local*: $\kappa_\gamma(s)$ depends on γ only in a neighborhood of s.

The second is commonly encountered as the geometric definition of curvature:

9. The radius of the *osculating circle* at $\gamma(s)$ is $\frac{1}{|\kappa(s)|}$. The osculating circle is the unique (possibly degenerate) circle that agrees with γ to second order at $\gamma(s)$.

Now we move to some less-obvious properties of curvature. Notice that a circle of radius r has curvature $\frac{1}{r}$ and circumference $2\pi r$, so that its total curvature integrated along the curve is 2π. Reverse the orientation, and you get total curvature -2π. Concatenate two circle, so that the circle "winds around twice," and you get 4π. If you deform the circle slightly, to make it an ellipse, you increase the curvature near the major axis and decrease it near the minor axis, suggesting that perhaps the total curvature stays the same. In fact,

Theorem 2.1 *(Whitney-Graustein) ("winding number theorem") For closed curves γ,*

$$\int_0^L \kappa(s)\, ds = 2\pi n, \quad n \in \mathbb{Z}.$$

Proof We can write $T(s)$ in coordinates as $(T^x(s), T^y(s))$, and furthermore identify $T(s)$ with a single complex number $\tau(s) = T^x(s) + iT^y(s)$. Notice that J acts exactly as multiplication by i: $JT \cong i\tau$. Property zero then becomes

$$\tau'(s) = -\kappa(s) i \tau(s).$$

This is an elementary ODE with solution

$$\tau(s) = A \exp\left(-i \int_0^s \kappa(t)\, dt\right)$$

and $A = \tau(0)$. Since γ is closed and T is unit-length, $\tau(L) = \tau(0) \neq 0$ and

$$1 = \cos\left(\int_0^L \kappa(s)\, ds\right) - i \sin\left(\int_0^L \kappa(s)\, ds\right)$$

which holds only when $\int_0^L \kappa(s)\, ds$ is an integer multiple of 2π. ∎

Another fact about curvature is that it is the "gradient of arc length," a property used all of the time when studying the shape of soap films and fluid interfaces. To make this idea more precise, recall from elementary Calculus that we define the directional derivative $(D_v f)(x)$ of an ordinary function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction v by the limit

$$\left(\frac{d}{dt} f(x + tv)\right)\Big|_{t\to 0}.$$

The gradient $(\nabla f)(x)$ is then *defined* as the unique vector with the property that $\nabla f \cdot v = D_v f$ for all directions v.

Similarly, every smooth vector field δ along γ is a "direction" along which we could vary γ to get a new curve $\gamma + t\delta$. As we smoothly change t, we smoothly change γ, and so we smoothly change the arc length of γ: the directional derivative

$$(D_\delta L)(\gamma) = \left(\frac{d}{dt} L(\gamma + t\delta)\right)\Big|_{t\to 0}$$

is well-defined. We define the *gradient* ∇L to be the vector field along γ that satisfies

$$\langle \nabla L, \delta \rangle = D_\delta L$$

where $\langle v, w\rangle = \int_0^L v \cdot w\, ds$ is the L^2 inner product along γ. (It should be checked that such a gradient always exists and is unique, as is the case for gradients of smooth functions, but we won't worry about that here.)

Theorem 2.2 *For a closed curve $\gamma(s)$, $\nabla_\gamma L = \kappa(s)N(s)$.*

Proof Pick an arbitrary vector field δ. Since γ is closed, we restrict to vector fields continuous across the curve endpoints, $\delta(0) = \delta(L)$. We can compute explicitly the directional derivative $D_\delta L$. First,

$$\begin{aligned} L(\gamma + t\delta) &= \int_0^{L(\gamma)} \|\gamma' + t\delta'\|\, ds \\ \frac{d}{dt} L(\gamma + t\delta) &= \int_0^{L(\gamma)} \frac{\gamma' + t\delta'}{\|\gamma' + t\delta'\|}\delta'\, ds \\ D_\delta L &= \int_0^{L(\gamma)} \gamma' \cdot \delta'\, ds. \end{aligned}$$

Integrating by parts, and using the fact that $\gamma'(0) = \gamma'(L)$ and $\delta(0) = \delta(L)$ since γ is closed,

$$D_\delta L = \gamma' \cdot \delta \Big|_0^L - \int_0^L \gamma'' \cdot \delta\, ds = \int_0^L \kappa(s)N(s) \cdot \delta(s)\, ds = \langle \kappa N, \delta\rangle.$$

Since the above holds for arbitrary δ, $\nabla L = \kappa N$. ∎

We will look at one last property of curvature. Suppose γ is closed and simple (injective on $[0, L)$: the curve does not self-intersect). Then γ encloses a well-defined region Ω_0 of the plane. Suppose further that γ is oriented so that its normal vectors $N(s)$ point outwardly from Ω_0. We can "inflate" the region by flowing the boundary along γ's normal vector, to get a new boundary curve $\gamma_\epsilon(s) = \gamma(s) + \epsilon N(s)$ and new enclosed region Ω_ϵ. How does the area of the new region $A(\Omega_\epsilon)$ compare to the area of the original region $A(\Omega_0)$?

Clearly, the two areas are equal to each to zeroth order, and to first order, the area of the inflated region increases proportionally to the arc length of the original curve:

$$A(\Omega_\epsilon) = A(\Omega_0) + \epsilon L + O(\epsilon^2).$$

For a circle of radius r, $A(\Omega_\epsilon) = \pi(r+\epsilon)^2 = \pi r^2 + 2\pi r\epsilon + \pi\epsilon^2$, and we see a quadratic correction term, but nothing third order or higher. In fact,

Theorem 2.3 *For ϵ sufficiently small and γ closed, simple, and oriented clockwise, $A(\Omega_\epsilon) = A(\Omega_0) + \epsilon L + \frac{\epsilon^2}{2}\int_0^L \kappa(s)\, ds$. By the winding number theorem, the quadratic term is always equal to 2π, but for reasons that will become clear in the future we will keep the term in its unevaluated form.*

Proof The area of the inflated region is given by $\int_{\Omega_\epsilon} 1\, dA$. By Stokes' theorem, we have

$$\int_{\Omega_\epsilon} 1\, dA = \int_{\Omega_\epsilon} \frac{1}{2}\nabla \cdot (x, y)\, dA = \int_{\partial\Omega_\epsilon} \frac{1}{2}(x, y) \cdot N_\epsilon\, ds_\epsilon = \int_0^L \frac{1}{2}\gamma_\epsilon(s) \cdot N_\epsilon(s)\, ds_\epsilon,$$

where $N_\epsilon(s)$ is the normal of the inflated curve γ_ϵ and ds_ϵ is its length element (notice that γ_ϵ is *not* arc-length parameterized.) γ_ϵ has the same normal as γ, and

$$ds_\epsilon = \|\gamma' + \epsilon N'\| ds = \|T + \epsilon JT'\| ds = \|T - \epsilon\kappa JN\| ds = \|T - \epsilon\kappa J^2 T\| = 1 + \epsilon\kappa,$$

since $J^2 = -I$. (Here we assume that $\epsilon < \kappa$, so that we do not develop any cusps during inflation.) Therefore

$$A(\Omega_\epsilon) = \int_0^L \frac{1}{2}(\gamma(s) + \epsilon N(s)) \cdot N(s)(1 + \epsilon\kappa(s))\, ds.$$

Gathering terms by power of ϵ, and integrating by parts,

$$\begin{aligned} A(\Omega_\epsilon) &= \int_0^L \frac{1}{2}\gamma \cdot N\, ds + \epsilon \int_0^L \frac{1}{2}\, ds + \epsilon \int_0^L \frac{1}{2}\gamma \cdot N\kappa\, ds + \frac{\epsilon^2}{2}\int_0^L \kappa(s)\, ds \\ &= \int_{\Omega_0} 1\, dA + \epsilon\frac{L}{2} + \epsilon \int_0^L \frac{1}{2}\gamma \cdot -\gamma''\, ds + \frac{\epsilon^2}{2}\int_0^L \kappa(s)\, ds \\ &= A(\Omega_0) + \epsilon\frac{L}{2} + \epsilon\left[-\frac{1}{2}\gamma\cdot\gamma'\Big|_0^L + \frac{1}{2}\int_0^L \gamma'\cdot\gamma'\, ds\right] + \frac{\epsilon^2}{2}\int_0^L \kappa(s)\, ds \\ &= A(\Omega_0) + \epsilon L + \frac{\epsilon^2}{2}\int_0^L \kappa(s)\, ds. \quad \blacksquare \end{aligned}$$

3 Discrete Curve Geometry

We will now develop a theory of curve geometry that *parallels* the smooth theory reviewed above. First, we need a discrete curve, which we will take to be a sequence of points in the plane connected by straight lines. Formally, a *discrete curve* Γ is an ordered tuple $(v_0, v_1, \ldots, v_{n-1}) \in \mathbb{R}^{2n}$ of *vertices* $v_i \in \mathbb{R}^2$, with $v_{i+1} \neq v_i$ for all i. Between each consecutive pair of vertices, we have edges represented by edge vectors $e_{i+1/2} = v_{i+1} - v_i$. In this chapter, we will concentrate on closed discrete curves, with a final edge $e_{n-1/2}$ joining the last vertex to the first, and to avoid cumberson notation will take all vertex indices modulo n, so that for instance we can write v_n for v_0.

What geometric quantities can we measure on Γ? The concept of arc length carries over directly: in particular, the length of Γ between vertices v_i and v_{i+1} is simply $\|e_{i+1/2}\|$, and the total arc length the sum

$$L = \sum_{i=0}^{n-1} \|e_{i+1/2}\|.$$

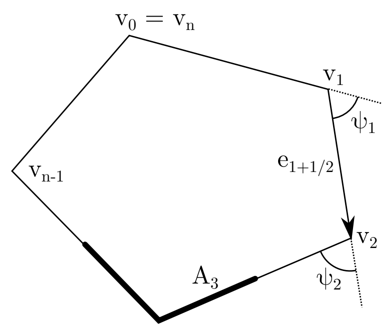

Figure 1: Notation for the various elements of Γ.

The edge vectors give us a natural tangent vector $T_{i+1/2}$ along each edge equal to $\frac{e_{i+1/2}}{\|e_{i+1/2}\|}$, and normal vector $N_{i+1/2} = JT_{i+1/2}$. If we think of the curve as a piecewise smooth, piecewise affine curve, these definitions of tangent vector and normal vector make sense almost everywhere, but certainly not at the vertices. Is there a sensible notion of normal vector at a vertex v_i, for instance? This is a surprisingly subtle question that will be revisited in the exercises.

Even more problematically, the above tangent and normal vectors are piecewise constant, so that applying the smooth definition of curvature to them directly is nonsensical. How, then, can we compute the curvature of Γ? There are several typical approaches.

- You don't consider Γ to be the "real" curve. Instead, it is one element of a family of curves Γ^i converging to a smooth limit curve γ, with some refinement rule $\Gamma^i \to \Gamma^{i+1}$ specifying how to construct this family. Perform all calculations on γ and call *that* the curvature of Γ.

  There are several downsides to this approach. First, the refinement rule must be specified, and there are *many* possible rules that might be chosen (four-point subdivision, B-spline subdivision, etc.) each of which will give a different limit curve. It doesn't make sense to think of Γ as representing some "platonic ideal" curve γ independent of specifying the refinement rule! Second, for most rules you might pick, actually computing γ can be very difficult, and measuring quantities such as curvature on γ can give counterintuitive results: for instance, it is easy to construct a family of discrete curves of arc length 4 that converge to a unit circle of circumference 2π.

- You don't consider Γ to be the "real" curve. Instead, it represent one element $\gamma(s) = \sum_i v_i b_i(s)$ of some finite-dimensional subspace of smooth curves spanned by smooth basis elements $b_i(s)$. This is the key idea behind the method of finite elements, and is incredibly powerful – when the subspace is chosen correctly. This is not the approach that we will take, but a lot of what we do could be suitably interpreted to look like first-order finite elements.

- You *do* consider Γ to be the real curve! Instead of taking the differential equation that defines curvature and trying to discretize it directly, we can instead look at the *symmetries, properties, and relationships* that κ obeys for smooth curves, and use those to *construct* an analogous quantity with analogous structure on Γ. This is the philosophy at the heart of discrete differential geometry, and in the remainder of this chapter we will see it in practice as we try to construct a discrete curvature for plane curves.

First, we will need a bit more machinery. For a smooth curve, curvature is a scalar function over that curve – we need an analogous concept of a discrete function over Γ. In particular, where a smooth function assigns a scalar to every point on γ, a discrete function assigns a scalar to every vertex of Γ. Let $\Omega^0(\Gamma) \cong \mathbb{R}^n$ be the space of discrete functions on Γ, where each element $F \in \mathbb{R}^n$ of Ω^0 is an assignment of a scalar F_i to each vertex v_i of Γ.

We also need an inner product on $\Omega^0(\Gamma)$. For smooth functions, we have the L^2 norm $\langle f, g\rangle = \int_0^L fg\,ds$; besides the usual properties of an inner product, notice that this one is *local*, depending only on f, g, and γ in the neighborhood of the same point $\gamma(s)$. Notice also that $\langle 1, 1\rangle = L$.

What is the discrete inner product $\langle F, G\rangle$? First, observe that every possible inner product can be written in the form $\langle F, G\rangle = F^T AG$, where the matrix A depends only on Γ and not on F or G. If we want a property analogous to the locality of the smooth inner product, we want A diagonal, with the ith diagonal entry A_i depending only on the part of Γ near v_i. We also want $\langle 1, 1\rangle = L$. These two desiderata lead to one natural choice: assign to each vertex v_i a discrete length element $A_i = \frac{\|e_{i-1/2}\|+\|e_{i+1/2}\|}{2}$, so that

$$\langle F, G\rangle = \sum_{i=0}^{n-1} A_i F_i G_i.$$

3.1 Properties of Discrete Curvature

We want to define a discrete curvature $k_i \in \Omega^0(\Gamma)$. We can start by browsing the list of properties of κ and applying those, axiomatically, to define k_i. Some properties are not immediately useful: for instance, property zero cannot be used since we cannot differentiate $T_{i+1/2}$, and it is not clear how to define an osculating circle to Γ.

However, κ obeys some useful symmetries. It is an isometry invariant; in order for k_i to be isometry invariant, it must remain unchanged if we translate or rotate Γ. This means that k_i cannot depend arbitrarily on the vertices v_i, but instead can depend only on quantities that are themselves invariant under plane isometries: lengths $\|e_{i+1/2}\|$ and angles. We need to be careful how we define these angles with respect to the orientation of Γ: for instance we can work with the *clockwise turning angle* ψ_i between $e_{i-1/2}$ and $e_{i+1/2}$.

So k_i depends on edge lengths and angles. But not any edge lengths and angles: locality of curvature imposes that k_i can only depend on those lengths and angles near v_i – in particular, $\|e_{i-1/2}\|, \|e_{i+1/2}\|$, and ψ_i. The behavior of $\kappa(s)$ under reflection and parameterization-reversal also give us some symmetries k_i must satisfy:

$$\begin{aligned} k_i(\|e_{i-1/2}\|, \|e_{i+1/2}\|, \psi_i) &= k_i(\|e_{i+1/2}\|, \|e_{i-1/2}\|, \psi_i) \\ &= -k_i(\|e_{i-1/2}\|, \|e_{i+1/2}\|, -\psi_i). \end{aligned}$$

What else? We have several other smooth properties of curvature, including the three deeper properties: the winding number theorem, the curvature normal as the gradient of arc length, and the behavior of area under inflation. Let's look at each of these in turn.

3.2 Discrete Winding Number Theorem

For smooth closed curves $\gamma(s)$, we had that $\int_0^L \kappa(s)\,ds = 2\pi n$ for some integer n. We can interpret that left-hand side as an inner product:

$$\langle \kappa, 1\rangle = 2\pi n.$$

We can use this identity as a guiding principle for *defining* discrete curvature k_i. If we do so, we will end up with a formula for discrete curvature that obeys the winding number theorem – not just approximately, and not just in the limit of refinement, but exactly.

Recall from elementary geometry that the turning angles of a simple polygon add up to $\pm 2\pi$. This is not hard to prove – by induction on the number of sides, for instance – but it's also intuitively clear: translate all turning angles so that they lie around a common point, and they add up to a complete rotation. It follows that all polygons, including those that self-intersect, have total turning angle $\sum \psi_i = 2\pi n$: to see this, induct on the number of self-intersections, and cut the polygon into two separate polygons at a point of self-intersection.

One choice of discrete curvature that would satisfy the winding number theorem, then, is to set

$$\langle k_i, 1\rangle = \sum_{i=0}^{n-1} \psi_i.$$

This gives us $\sum_{i=0}^{n-1} A_i k_i = \sum_{i=0}^{n-1} \psi_i$, and since the only turning angle that k_i can depend on is ψ_i,

$$k_i = \frac{2\psi_i}{\|e_{i-1/2}\| + \|e_{i+1/2}\|}. \tag{1}$$

We have a formula for discrete curvature!

3.3 Discrete Gradient of Arc Length

Instead of looking at the winding number theorem, we could instead have constructed discrete curvature starting from the desideratum that the discrete curvature obey a discrete analogue of the smooth fact that $\nabla_\gamma L = \kappa(s)N(s)$.

As in the smooth case, we can define the gradient of arc length by way of the directional derivative. Consider an arbitrary variation $\delta \in \Omega^0(\Gamma)^2$, where $\delta_i \in \mathbb{R}^2$ is a displacement of the vertex v_i of Γ. Then for any scalar t, we have a discrete curve $\Gamma + t\delta$ with vertices $v_i + t\delta_i$. Although Γ is a discrete curve and δ a discrete function, the total arc length $L(\Gamma + t\delta)$ is *smooth* as a function of t, and so the derivative $D_\delta L$ of the arc length of Γ in the δ direction is well-defined.

The discrete gradient $\nabla L \in \Omega^0(\Gamma) \times \Omega^0(\Gamma)$ is then the discrete vector-valued function (if it exists) satisfying

$$\langle \nabla L, \delta\rangle = D_\delta L \tag{2}$$

for every discrete variation δ. Earlier we defined an inner product for scalar-valued discrete functions on Γ; for vector-valued functions we extend this definition in the straightforward way:

$$\langle \nabla L, \delta\rangle = \sum_{i=0}^{n-1} A_i (\nabla L)_i \cdot \delta_i.$$

We want equation 2 to hold for *every* variation δ, so in particular it must hold for the variation

$$\delta_i = \begin{cases} 0, & i \neq j \\ w, & i = j \end{cases}$$

where j is some vertex of Γ and some $w \in \mathbb{R}^2$. This variation corresponds to moving one vertex of Γ, and leaving all other vertices fixed. Plugging this particular variation into equation 2 yields

$$\begin{aligned}
\sum_{i=0}^{n-1} A_i (\nabla L)_i \cdot \delta_i = A_j (\nabla L)_j \cdot w &= \left(\frac{d}{dt} L(\Gamma + t\delta) \right) \Bigg|_{t \to 0} \\
&= \left(\frac{d}{dt} \sum_{i=0}^{n-1} \| v_{j+1} + t\delta_{j+1} - v_j - t\delta_j \| \right) \Bigg|_{t \to 0} \\
&= \left(\frac{d}{dt} \left(\| v_j + tw - v_{j-1} \| + \| v_{j+1} - v_j - tw \| \right) \right) \Bigg|_{t \to 0} \\
&= \left(\frac{v_j + tw - v_{j-1}}{\| v_j + tw - v_{j-1} \|} \cdot w - \frac{v_{j+1} - v_j - tw}{\| v_{j+1} - v_j - tw \|} \cdot w \right) \Bigg|_{t \to 0} \\
&= \left(\frac{e_{j-1/2}}{\| e_{j-1/2} \|} - \frac{e_{j+1/2}}{\| e_{j+1/2} \|} \right) \cdot w.
\end{aligned}$$

This equality must hold for any choice of w, so

$$(\nabla L)_j = \frac{1}{A_j} \left(\frac{e_{j-1/2}}{\| e_{j-1/2} \|} - \frac{e_{j+1/2}}{\| e_{j+1/2} \|} \right).$$

The direction of this discrete vector field gives us a notion of normal vectors at vertices N_i. The magnitude gives us another way of deriving discrete curvature:

$$\begin{aligned}
|k_i| &= \| (\nabla L)_i \| \\
&= \frac{1}{A_i} \sqrt{2 - 2 \frac{e_{j-1/2}}{\| e_{j-1/2} \|} \cdot \frac{e_{j+1/2}}{\| e_{j+1/2} \|}} \\
&= \frac{1}{A_i} \sqrt{2 - 2 \cos \psi_i} \\
&= \frac{2}{A_i} \left| \sin \frac{\psi_i}{2} \right|.
\end{aligned}$$

Taking into account the sign of k_i gives us

$$k_i = \frac{4 \sin \frac{\psi_i}{2}}{\| e_{i-1/2} \| + \| e_{i+1/2} \|}. \tag{3}$$

Notice that this does *not* agree with the formula we got from the winding number theorem, in equation (1)! In fact, it is easy to see that this formula does not satisfy the winding number theorem. Consider a regular n-gon, with turning angle $\psi_i = \frac{2\pi}{n}$. For n large, ψ_i is small, and $0 < \sin \frac{\psi_i}{2} < \frac{\psi_i}{2}$. This gives us that

$$0 < \langle k, 1 \rangle = \sum_{i=0}^{n-1} 2 \sin \frac{\psi_i}{2} = 2n \sin \frac{2\pi}{2n} < 2\pi,$$

violating the turning angle theorem.

3.4 Discrete Area Inflation

We now have two different formulas for curvature. Recall that for smooth closed injective curves,

$$A(\Omega_\epsilon) = A(\Omega_0) + \epsilon L + \frac{\epsilon^2}{2} \int_0^L \kappa(s)\, ds.$$

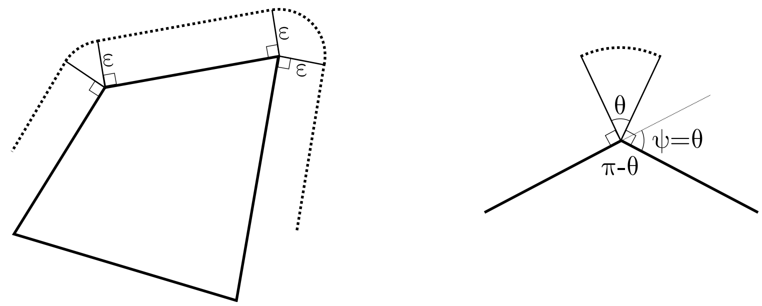

Figure 2: *Left:* Inflation of Γ by Mikowski-summing with a circle of radius ϵ. *Right*: At each vertex v_i, the angle of the inflated circle sector is equal to the turning angle ψ_i.

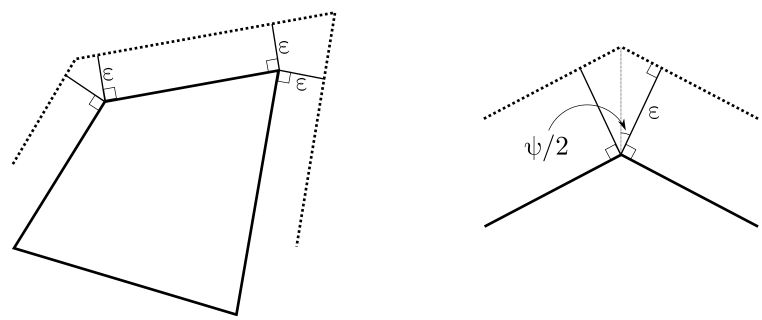

Figure 3: *Left:* Inflation of Γ by parallel lines distance ϵ from Γ's edges. *Right*: The area of each kite at vertex v_i is equal to $\epsilon^2 \tan \frac{\psi_i}{2}$.

Could a disretization of this formula give us yet another notion of discrete curvature?

For a closed discrete curve Γ with no self-intersections, it is easy to compute the area of the enclosed polygon Ω_0. But how do we define the "inflated" region Ω_ϵ? There are several reasonable choices.

First, we could take the *Mikowski sum* of Ω_0 with a ball of radius ϵ: in other words, sweep a ball of radius ϵ around Γ and take the outer boundary of the resulting region as the boundary of Ω_ϵ. See figure 2.

We can compute the area of Ω_ϵ by cutting it up into easily-managed pieces. In the center, we have the area of the original region $A(\Omega_0)$. Along each edge, we have rectangular pieces of area $\epsilon\|e_{i+1/2}\|$. At every vertex, we have a sector of a circle of radius ϵ. The area of each sector is $\pi\epsilon^2 \cdot \frac{\theta}{2\pi} = \frac{\epsilon^2}{2}\theta$, where θ is the sector angle; this angle is equal to the vertex's turning angle ψ_i (see figure 2.)

We therefore have that

$$A(\Omega_\epsilon) = A(\Omega_0) + \epsilon L + \frac{\epsilon^2}{2}\sum_{i=0}^{n-1}\psi_i,$$

and in order to match the smooth formula we want $k_i A_i = \psi_i$, recovering the curvature formula in equation (1).

But we could have chosen a different way of inflating Ω_0: for instance, we could replace all edges with parallel edges a distance ϵ away, with kites instead of sectors at the corners (see figure 3). The area of the kites is then $\epsilon^2 \tan \frac{\psi_i}{2}$, giving us the formula

$$A(\Omega_\epsilon) = A(\Omega_0) + \epsilon L + \frac{\epsilon^2}{2}\sum_{i=0}^{n-1} 2\tan\frac{\psi_i}{2},$$

Formula	Basic Symmetries (2–4; 8)	Diverges at Kinks (7)	Winding Number Theorem (10)	Gradient of Length (11)	Inflation Theorem (12)
$\frac{2\psi_i}{\|e_{i-1/2}\|+\|e_{i+1/2}\|}$	Yes	No	Yes	No	Yes
$\frac{4\sin\frac{\psi_i}{2}}{\|e_{i-1/2}\|+\|e_{i+1/2}\|}$	Yes	No	No	Yes	Maybe?
$\frac{4\tan\frac{\psi_i}{2}}{\|e_{i-1/2}\|+\|e_{i+1/2}\|}$	Yes	Yes	No	No	Yes

Table 1: List of formula for discrete curvature, and summary of some of their properties (numbers in parentheses refer to properties of the smooth curvature $\kappa(s)$.) Notice that none of the formulas satisfy all properties. Although we didn't derive the sine formula from the inflation theorem, we haven't shown that there isn't some interpretation of inflation that does yield that formula.

and inducing yet another definition of discrete curvature:

$$k_i = \frac{4\tan\frac{\psi_i}{2}}{\|e_{i-1/2}\| + \|e_{i+1/2}\|}. \tag{4}$$

As with the formula in equation (3), this discrete curvature violates the turning angle theorem: again, it suffices to check a regular n-gon, and employ the fact that for small angles θ, $\tan\theta > \theta$.

One final remark about the discrete area inflation theorem: all of these calculations have implicitly assumed that Ω is convex – otherwise, the diagrams in the above figures are incorrect. It is interesting to consider what happens when Ω has vertices of negative curvature.

3.5 Summary

We now have three different formulas for discrete curvature. Table 1 summarizes these formulas and some their properties. There are several key points to take away from these calcuations:

- It is possible to build up a theory of discrete differential geometry that parallels smooth differential geometry. The discrete analogues of geometric measures like arc length, gradient, curvature, etc. can be made to respect the symmetries and relationships of the smooth measures *exactly*: not just approximately, and not just in some limit of refinement.
- There is no unique choice of how to discretize geometry. Above we saw three equally principled ways of defining discrete curvature of curves in the plane, for instance. DDG is all about paying close attention to what choices lead to what discretizations, and what the consequences are of these choices.
- There is no free lunch. None of the three formulas for curvature respected *all* of the properties that are satisfied by smooth curvature – each respects a different subset. As a practical consequence, when choosing a discretization for computation, one must be aware of the alternatives and their pros and cons, and select a discretization best suited for the particular task.

3.6 Postscript on Convergence

Suppose you have a sequence of discrete curves Γ^i converging to a smooth curve γ. For which of the formulas in table 1 does discrete curvature converge to smooth curvature? Note that this question is imprecise – what do we mean by convergence of the curves? Convergence of the curvature? If we ask only for pointwise convergence of Γ^i to γ, the answer is clearly "none of them": it is easy to construct a family of increasingly oscillatory discrete curves converging to a unit circle.

It can be shown that all of the formulas *do* converge to $\kappa(s)$ when Γ^i converges to γ under an appropriate Sobolev distance

$$d(\Gamma^i, \gamma) = \int \|\Gamma^i(s) - \gamma(s)\| + \|T_j(s) - \gamma'(s)\|\, ds$$

that takes into account both the positions and tangents of Γ^i. We won't concern ourselves too much with details about convergence results, which tend to be subtle and technical, in this course – all of the discrete formulas we will see in this course do converge to their smooth counterparts under "nice enough" refinement of the discrete geometry, which I will occasionally make more precise in passing.