An unavoidable case analysis

Archiving old manuscripts, I found at the end of EWD766 "An educational stupidity" of more than 20 years ago the following exercise:

"Prove that none of the decimal numbers 1001, 1001001, 1001001001, 1001001001001, ... is prime." [It is not clear why I underlined "none". EWD]

Here is a proof. Denoting by "k*<string>*" the concatenation of k copies of the digit string enclosed, we deal with the decimal numbers

k* 001* for $k \geq 2$

(i) If k mod $3 = 0$, the number is, according to the traditional 3-test, divisible by 3 because the sum of its decimal digits, which equals k, is divisible by 3.

(ii) If k mod $3 \neq 0$, we see by generalizing the traditional 9-test to the k*9*-test, that the number reduced modulo k*9* equals k*1*. Hence the number is divisible by k*1* (because k*9* is). (Note that this observation does not exclude primality for $k=1$.)

In argument (i), the crux is the validity of the 3-test, which is valid whenever base mod 3 = 1, a condition that base 10 (= ten) satisfies. The conclusion has nothing to do with the accident that the length of the repeated string happens to be 3: for k mod $3 = 0$, also ${}^{k*}00001^*$ is divisible by 3.

In argument (ii) we generalized the 9-test because the problem was about decimal numbers, but for any base B $(B \geq 2)$ there is a $(B-1)$-test, and ${}^{k*}1^*$ divides ${}^{k*}(B-1)^*$. The conclusion that the number reduced modulo ${}^{k*}(B-1)^*$ yields ${}^{k*}1^*$, however, depends on the fact that k has no factor in common with the length of the iterated string. Argument (ii) is base independent: interpreting ${}^{*k}001^*$ and ${}^{*k}1^*$ in binary, we find for instance that for $k=4$ and $k=5$, 585 is divisible by 15 and 4681 by 31.

Argument (i) relies on a relation between k and the base of the number system, (ii) on a relation between k and the length of the iterated string.

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