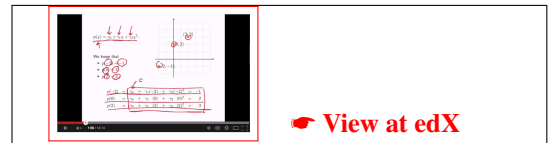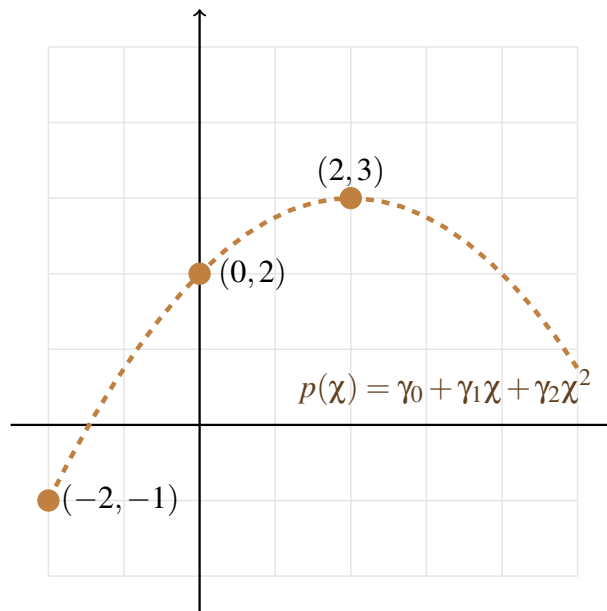# 9

Vector Spaces

9.1 Opening Remarks

9.1.1 Solvable or not solvable, that's the question

☛ **View at edX**

Consider the picture

depicting three points in $\mathbb{R}^2$ and a quadratic polynomial (polynomial of degree two) that passes through those points. We say that this polynomial *interpolates* these points. Let's denote the polynomial by

$$p(\chi) = \gamma_0 + \gamma_1 \chi + \gamma_2 \chi^2.$$

How can we find the coefficients γ_0, γ_1, and γ_2 of this polynomial? We know that $p(-2) = -1$, $p(0) = 2$, and $p(2) = 3$. Hence

$$
\begin{aligned}
p(-2) &= \gamma_0 + \gamma_1(-2) + \gamma_2(-2)^2 &= -1 \\
p(0) &= \gamma_0 + \gamma_1(0) + \gamma_2(0)^2 &= 2 \\
p(2) &= \gamma_0 + \gamma_1(2) + \gamma_2(2)^2 &= 3
\end{aligned}
$$

In matrix notation we can write this as

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$
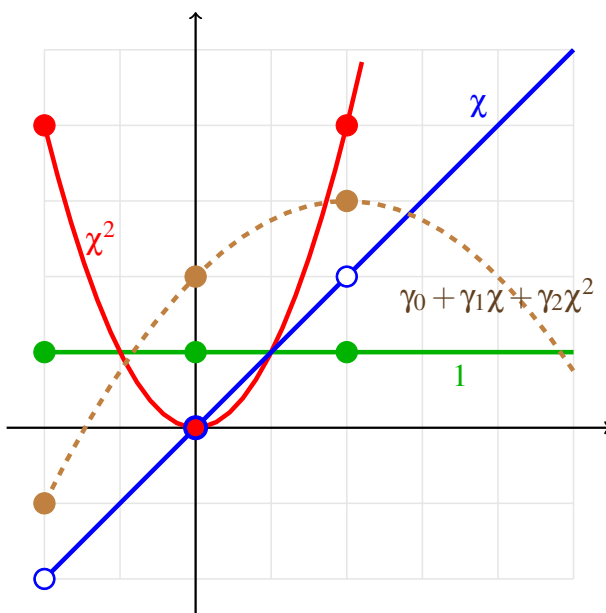
By now you have learned a number of techniques to solve this linear system, yielding

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}$$

so that

$$p(\chi) = 2 + \chi - \frac{1}{4}\chi^2.$$

Now, let's look at this problem a little differently. $p(\chi)$ is a linear combination (a word you now understand well) of the polynomials $p_0(\chi) = 1$, $p_1(\chi) = \chi$, and $p_2(\chi) = \chi^2$. These basic polynomials are called "parent functions".

Now, notice that

$$\begin{pmatrix} p(-2) \\ p(0) \\ p(2) \end{pmatrix} = \begin{pmatrix} \gamma_0 & + & \gamma_1(-2) & + & \gamma_2(-2)^2 \\ \gamma_0 & + & \gamma_1(0) & + & \gamma_2(0)^2 \\ \gamma_0 & + & \gamma_1(2) & + & \gamma_2(2)^2 \end{pmatrix}$$

$$= \gamma_0 \begin{pmatrix} p_0(-2) \\ p_0(0) \\ p_0(2) \end{pmatrix} + \gamma_1 \begin{pmatrix} p_1(-2) \\ p_1(0) \\ p_1(2) \end{pmatrix} + \gamma_2 \begin{pmatrix} p_2(-2) \\ p_2(0) \\ p_2(2) \end{pmatrix}$$

$$= \gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} (-2)^2 \\ 0^2 \\ 2^2 \end{pmatrix}$$

$$= \gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}.$$

You need to think of the three vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$ as vectors that capture the polynomials p_0, p_1, and p_2

at the values -2, 0, and 2. Similarly, the vector $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ captures the polynomial p that interpolates the given points.
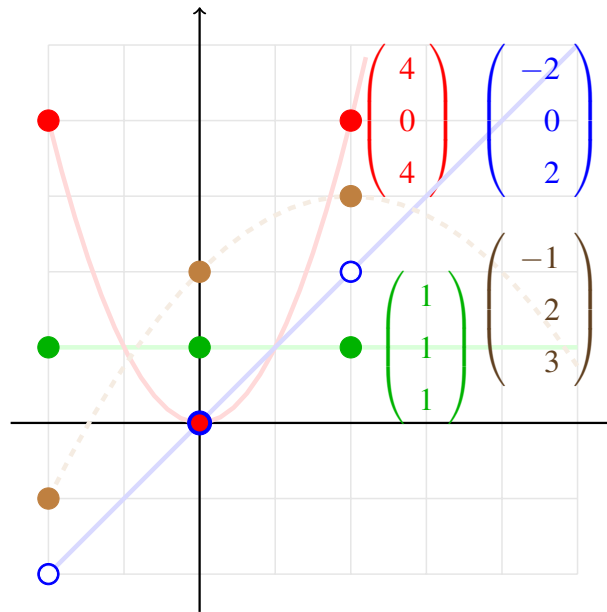
What we notice is that this last vector must equal a linear combination of the first three vectors:

$$\gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Again, this gives rise to the matrix equation

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

with the solution

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}.$$

The point is that one can think of finding the coefficients of a polynomial that interpolates points as either solving a system of linear equations that come from the constraint imposed by the fact that the polynomial must go through a given set of points, or as finding the linear combination of the vectors that represent the parent functions at given values so that this linear combination equals the vector that represents the polynomial that is to be found.
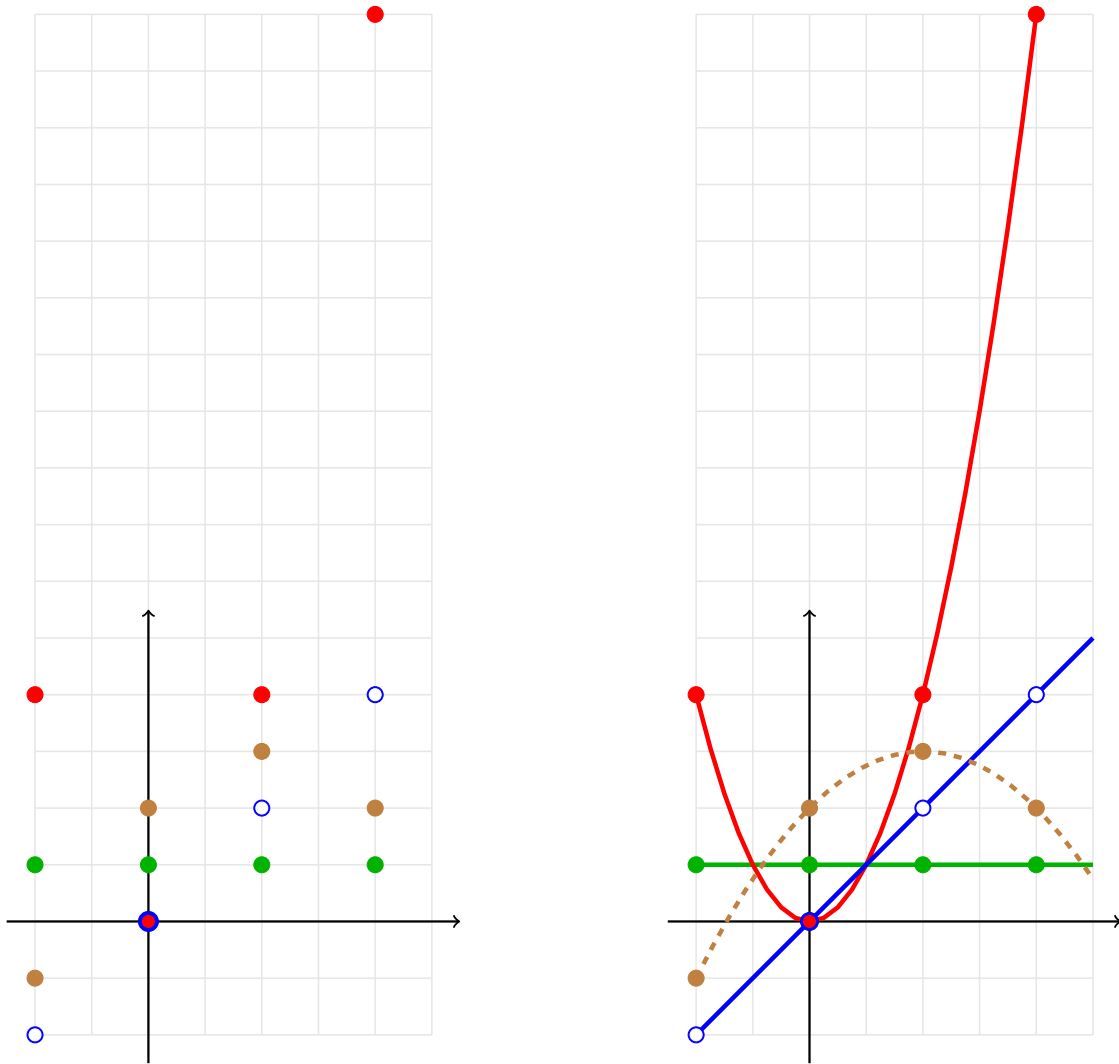
Figure 9.1: Interpolating with at second degree polynomial at $\chi = -2, 0, 2, 4$.

To be or not to be (solvable), that's the question

Next, consider the picture in Figure 9.1 (left), which accompanies the matrix equation

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 2 \end{pmatrix}.$$

Now, this equation is also solved by

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}.$$

The picture in Figure 9.1 (right) explains why: The new brown point that was added happens to lie on the overall quadratic polynomial $p(\chi)$.
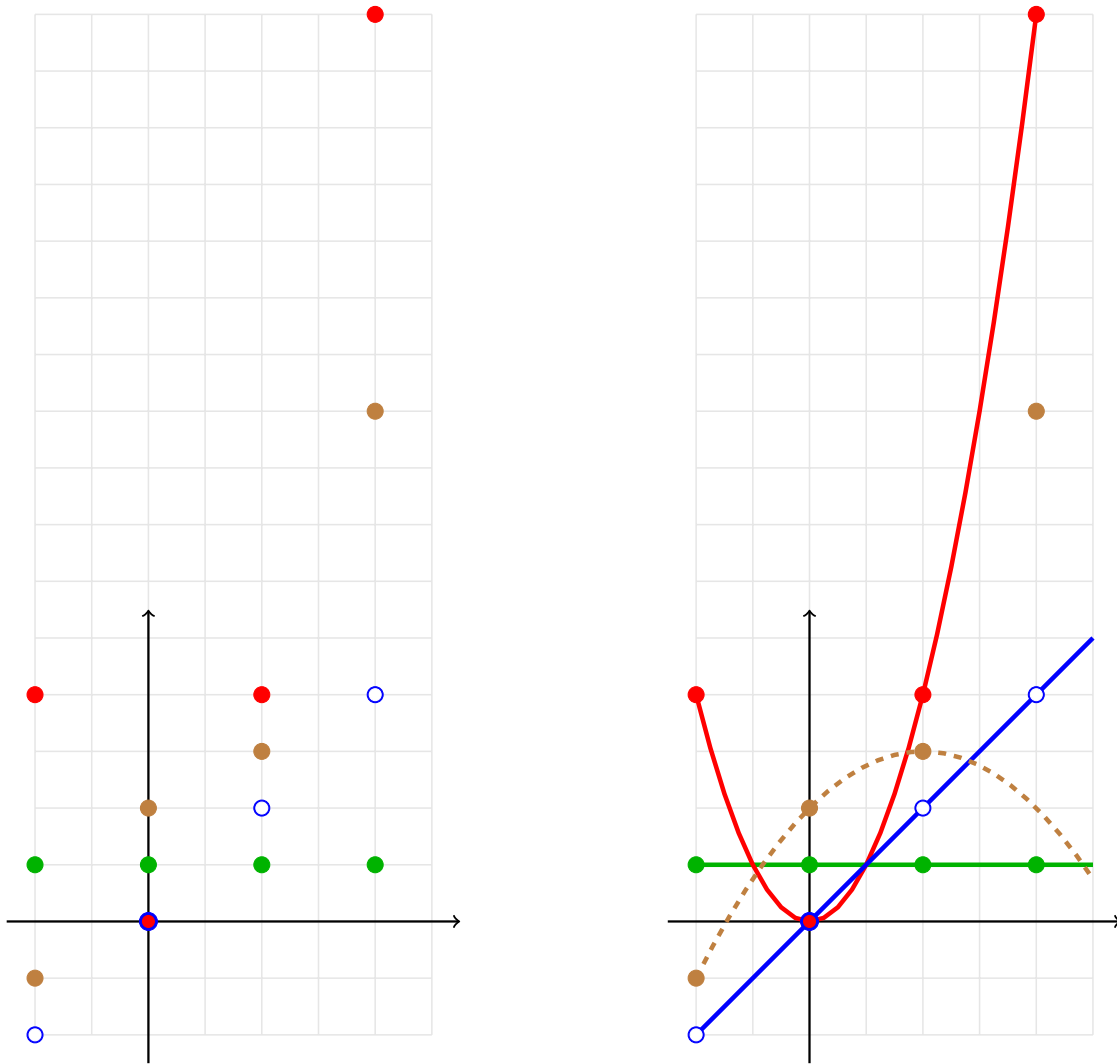
Figure 9.2: Interpolating with at second degree polynomial at $\chi = -2, 0, 2, 4$: when the fourth point doesn't fit.

Finally, consider the picture in Figure 9.2 (left) which accompanies the matrix equation

$$
\begin{pmatrix}
1 & -2 & 4 \\
1 & 0 & 0 \\
1 & 2 & 4 \\
1 & 4 & 16
\end{pmatrix}
\begin{pmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2
\end{pmatrix}
=
\begin{pmatrix}
-1 \\
2 \\
3 \\
9
\end{pmatrix}.
$$

It turns out that this matrix equation (system of linear equations) does not have a solution. The picture in Figure 9.2 (right) explains why: The new brown point that was added does not lie on the quadratic polynomial $p_2(\chi)$.

This week, you will learn that the system $Ax = b$ for an $m \times n$ matrix A sometimes has a unique solution, sometimes has no solution at all, and sometimes has an infinite number of solutions. Clearly, it does not suffice to only look at the matrix A. It is how the columns of A are related to the right-hand side vector that is key to understanding with which situation we are dealing. And the key to understanding how the columns of A are related to those right-hand sides for which $Ax = b$ has a solution is to understand a concept called vector spaces.

9.1.2 Outline

9.1.3 What you will learn

Upon completion of this unit, you should be able to

- Determine when systems do not have a unique solution and recognize the general solution for a system.

- Use and understand set notation.

- Determine if a given subset of $\mathbb{R}^n$ is a subspace.

- For simple examples, determine the null space and column space for a given matrix.

- Identify, apply, and prove simple properties of sets, vector spaces, subspaces, null spaces and column spaces.

- Recognize for simple examples when the span of two sets of vectors is the same.

- Determine when a set of vectors is linearly independent by exploiting special structures. For example, relate the rows of a matrix with the columns of its transpose to determine if the matrix has linearly independent rows.

- For simple examples, find a basis for a subspace and recognize that while the basis is not unique, the number of vectors in the basis is.

9.2 When Systems Don't Have a Unique Solution

9.2.1 When Solutions Are Not Unique

Up until this week, we looked at linear systems that had exactly one solution. The reason was that some variant of Gaussian elimination (with row exchanges, if necessary and/or Gauss-Jordan elimination) completed, which meant that there was exactly one solution.

What we will look at this week are linear systems that have either no solution or many solutions (indeed an infinite number).

Example 9.1 Consider

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Does $Ax = b_0$ have a solution? The answer is yes:

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}. \checkmark$$

But this is not the only solution:

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \checkmark$$

and

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}. \checkmark$$

Indeed, later we will see there are an infinite number of solutions!

Example 9.2 Consider

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}.$$

We will show that this equation does not have a solution in the next unit.

Homework 9.2.1.1 Evaluate

1. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} =$

2. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} =$

3. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} =$

Does the system $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$ have multiple solutions?   Yes/No

9.2.2   When Linear Systems Have No Solutions

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Consider

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}.$$

• Set this up as an appended system

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -2 & -3 & 4 & 3 \\ 4 & 3 & -2 & 4 \end{array} \right).$$

Now, start applying Gaussian elimination (with row exchanges).

• Use the first row to eliminate the coefficients in the first column below the diagonal:

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & 2 & 4 \end{array} \right).$$

• Use the second row to eliminate the coefficients in the second column below the diagonal:

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

- At this point, we have encountered a zero on the diagonal of the matrix that cannot be fixed by exchanging with rows below the row that has the zero on the diagonal.

Now we have a problem: The last line of the appended system represents

$$0 \times \chi_0 + 0 \times \chi_1 + 0 \times \chi_2 = 1,$$

or,

$$0 = 1$$

which is a contradiction. Thus, the original linear system represented three equations with three unknowns in which a contradiction was hidden. As a result this system does not have a solution.

> Anytime you execute Gaussian elimination (with row exchanges) or Gauss-Jordan (with row exchanges) and at some point encounter a row in the appended system that has zeroes to the left of the vertical bar and a nonzero to its right, the process fails and the system has no solution.

Homework 9.2.2.1 The system $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$ has no solution.

True/False

9.2.3 When Linear Systems Have Many Solutions

☛ **View at edX**

Now, let's learn how to find one solution to a system $Ax = b$ that has an infinite number of solutions. Not surprisingly, the process is remarkably like Gaussian elimination:

Consider again

$$A = \begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}.$$

Set this up as an appended systems

$$\begin{pmatrix} 2 & 2 & -2 & | & 0 \\ -2 & -3 & 4 & | & 3 \\ 4 & 3 & -2 & | & 3 \end{pmatrix} \tag{9.1}$$

Now, apply Gauss-Jordan elimination. (Well, something that closely resembles what we did before, anyway.)

- Use the first row to eliminate the coefficients in the first column below the diagonal:

$$\begin{pmatrix} 2 & 2 & -2 & | & 0 \\ 0 & -1 & 2 & | & 3 \\ 0 & -1 & 2 & | & 3 \end{pmatrix}.$$

- Use the second row to eliminate the coefficients in the second column below the diagonal and use the second row to eliminate the coefficients in the second column above the diagonal:

$$\begin{pmatrix} 2 & 0 & 2 & | & 6 \\ 0 & -1 & 2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

- Divide the first and second row by the diagonal element:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Now, what does this mean? Up until this point, we have not encountered a situation in which the system, upon completion of either Gaussian elimination or Gauss-Jordan elimination, an entire zero row. Notice that the difference between this situation and the situation of no solution in the previous section is that the entire row of the final appended system is zero, including the part to the right of the vertical bar.

So, let's translate the above back into a system of linear equations:

$$
\begin{array}{rcrcrcr}
\chi_0 & & & + & \chi_2 & = & 3 \\
& & \chi_1 & - & 2\chi_2 & = & -3 \\
& & & & 0 & = & 0
\end{array}
$$

Notice that we really have two equations and three unknowns, plus an equation that says that "$0 = 0$", which *is true*, but doesn't help much!

Two equations with three unknowns does not give us enough information to find a unique solution. What we are going to do is to make χ_2 a "free variable", meaning that it can take on any value in $\mathbb{R}$ and we will see how the "bound variables" χ_0 and χ_1 now depend on the free variable. To so so, we introduce β to capture this "any value" that χ_2 can take on. We introduce this as the third equation

$$
\begin{array}{rcrcrcr}
\chi_0 & & & + & \chi_2 & = & 3 \\
& & \chi_1 & - & 2\chi_2 & = & -3 \\
& & & & \chi_2 & = & \beta
\end{array}
$$

and then substitute β in for χ_2 in the other equations:

$$
\begin{array}{rcrcrcr}
\chi_0 & & & + & \beta & = & 3 \\
& & \chi_1 & - & 2\beta & = & -3 \\
& & & & \chi_2 & = & \beta
\end{array}
$$

Next, we bring the terms that involve β to the right

$$
\begin{array}{rcrcr}
\chi_0 & & & = & 3 & - & \beta \\
& \chi_1 & & = & -3 & + & 2\beta \\
& & \chi_2 & = & & & \beta
\end{array}
$$

Finally, we write this as vectors:

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{c} 3 \\ -3 \\ 0 \end{array}\right) + \beta \left(\begin{array}{c} -1 \\ 2 \\ 1 \end{array}\right)$$

We now claim that this captures *all* solutions of the system of linear equations. We will call this the *general* solution.

Let's check a few things:

- Let's multiply the original matrix times the first vector in the general solution:

$$\left(\begin{array}{ccc} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{array}\right)\left(\begin{array}{c} 3 \\ -3 \\ 0 \end{array}\right) = \left(\begin{array}{c} 0 \\ 3 \\ 3 \end{array}\right). \checkmark$$

Thus the first vector in the general solution is a solution to the linear system, corresponding to the choice $\beta = 0$. We will call this vector a *specific* solution and denote it by x_s. Notice that there are many (indeed an infinite number of) specific solutions for this problem.

- Next, let's multiply the original matrix times the second vector in the general solution, the one multiplied by β:

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \checkmark$$

And what about the other solutions that we saw two units ago? Well,

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}. \checkmark$$

and

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3/2 \\ 0 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \checkmark$$

But notice that these are among the infinite number of solutions that we identified:

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + (1) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3/2 \\ 0 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + (3/2) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

9.2.4 What is Going On?

Consider $Ax = b$ and assume that we have

- One solution to the system $Ax = b$, the specific solution we denote by x_s so that $Ax_s = b$.

- One solution to the system $Ax = 0$ that we denote by x_n so that $Ax_n = 0$.

Then

$$A(x_s + x_n)$$
$$= \quad < \text{Distribute } A >$$
$$Ax_s + Ax_n$$
$$= \quad < Ax_s = b \text{ and } Ax_n = 0 >$$
$$b + 0$$
$$= \quad < \text{algebra} >$$
$$b$$

So, $x_s + x_n$ is *also* a solution.

Now,

$$A(x_s + \beta x_n)$$

$$= \qquad < \text{Distribute } A >$$

$$Ax_s + A(\beta x_n)$$

$$= \qquad < \text{Constant can be brought out} >$$

$$Ax_s + \beta Ax_n$$

$$= \qquad < Ax_s = b \text{ and } Ax_n = 0 >$$

$$b + 0$$

$$= \qquad < \text{algebra} >$$

$$b$$

So $A(x_s + \beta x_n)$ is a solution for *every* $\beta \in \mathbb{R}$.

> Given a linear system $Ax = b$, the strategy is to first find a specific solution, x_s such that $Ax_s = b$. If this is clearly a unique solution (Gauss-Jordan completed successfully with no zero rows), then you are done. Otherwise, find vector(s) x_n such that $Ax_n = 0$ and use it (these) to specify the general solution.

We will make this procedure more precise later this week.

> **Homework 9.2.4.1** Let $Ax_s = b$, $Ax_{n_0} = 0$ and $Ax_{n_1} = 0$. Also, let $\beta_0, \beta_1 \in \mathbb{R}$. Then $A(x_s + \beta_0 x_{n_0} + \beta_1 x_{n_1}) = b$.
> Always/Sometimes/Never

9.2.5   Toward a Systematic Approach to Finding All Solutions

☛ **View at edX**

Let's focus on finding nontrivial solutions to $Ax = 0$, for the same example as in Unit 9.2.3. (The trivial solution to $Ax = 0$ is $x = 0$.)

Recall the example

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

which had the general solution

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

We will again show the steps of Gaussian elimination, except that this time we **also** solve

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Set both of these up as an appended systems

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -2 & -3 & 4 & 3 \\ 4 & 3 & -2 & 3 \end{array} \right) \qquad \left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -2 & -3 & 4 & 0 \\ 4 & 3 & -2 & 0 \end{array} \right)$$

- Use the first row to eliminate the coefficients in the first column below the diagonal:

$$
\left(\begin{array}{ccc|c}
\boxed{2} & 2 & -2 & 0 \\
0 & -1 & 2 & 3 \\
0 & -1 & 2 & 3
\end{array} \right)
\qquad
\left(\begin{array}{ccc|c}
\boxed{2} & 2 & -2 & 0 \\
0 & -1 & 2 & 0 \\
0 & -1 & 2 & 0
\end{array} \right).
$$

- Use the second row to eliminate the coefficients in the second column below the diagonal

$$
\left(\begin{array}{ccc|c}
\boxed{2} & 2 & -2 & 0 \\
0 & \boxed{-1} & 2 & 3 \\
0 & 0 & 0 & 0
\end{array} \right)
\qquad
\left(\begin{array}{ccc|c}
\boxed{2} & 2 & -2 & 0 \\
0 & \boxed{-1} & 2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
$$

Some terminology

The form of the transformed equations that we have now reached on the left is known as the **row-echelon form**. Let's examine it:

$$
\left(\begin{array}{ccc|c}
\boxed{2} & 2 & -2 & 0 \\
0 & \boxed{-1} & 2 & 3 \\
0 & 0 & 0 & 0
\end{array} \right)
$$

The boxed values are known as the *pivots*. In each row to the left of the vertical bar, the left-most nonzero element is the pivot for that row. Notice that the pivots in later rows appear to the right of the pivots in earlier rows.

Continuing on

- Use the second row to eliminate the coefficients in the second column above the diagonal:

$$
\left(\begin{array}{ccc|c}
\boxed{2} & 0 & 2 & 6 \\
0 & \boxed{-1} & 2 & 3 \\
0 & 0 & 0 & 0
\end{array} \right)
\qquad
\left(\begin{array}{ccc|c}
\boxed{2} & 0 & 2 & 0 \\
0 & \boxed{-1} & 2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
$$

In this way, all elements above pivots are eliminated. (Notice we could have done this as part of the previous step, as part of the Gauss-Jordan algorithm from Week 8. However, we broke this up into two parts to be able to introduce the term **row echelon form**, which is a term that some other instructors may expect you to know.)

- Divide the first and second row by the diagonal element to normalize the pivots:

$$
\left(\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 3 \\
0 & \boxed{1} & -2 & -3 \\
0 & 0 & 0 & 0
\end{array} \right)
\qquad
\left(\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 0 \\
0 & \boxed{1} & -2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
$$

Some more terminology

The form of the transformed equations that we have now reached on the left is known as the **reduced row-echelon form**. Let's examine it:

$$
\left(\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 3 \\
0 & \boxed{1} & -2 & -3 \\
0 & 0 & 0 & 0
\end{array} \right)
\qquad
\left(\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 0 \\
0 & \boxed{1} & -2 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
$$

In each row, the pivot is now equal to one. All elements above pivots have been zeroed.

Continuing on again

- Observe that there was no need to perform all the transformations with the appended system on the right. One could have simply applied them only to the appended system on the left. Then, to obtain the results on the right we simply set the right-hand side (the appended vector) equal to the zero vector.

So, let's translate the left appended system back into a system of linear systems:

$$
\begin{array}{ccccccc}
\chi_0 & & + & \chi_2 & = & 3 \\
& \chi_1 & - & 2\chi_2 & = & -3 \\
& & & 0 & = & 0
\end{array}
$$

As before, we have two equations and three unknowns, plus an equation that says that "$0 = 0$", which is *true*, but doesn't help much! We are going to find *one solution* (a specific solution), by choosing the free variable $\chi_2 = 0$. We can set it to equal anything, but zero is an easy value with which to compute. Substituting $\chi_2 = 0$ into the first two equations yields

$$
\begin{array}{ccccccc}
\chi_0 & & + & 0 & = & 3 \\
& \chi_1 & - & 2(0) & = & -3 \\
& & & 0 & = & 0
\end{array}
$$

We conclude that a specific solution is given by

$$
x_s = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}.
$$

Next, let's look for *one* non-trivial solution to $Ax = 0$ by translating the right appended system back into a system of linear equations:

$$
\begin{array}{ccccccc}
\chi_0 & & + & \chi_2 & = & 0 \\
& \chi_1 & - & 2\chi_2 & = & 0
\end{array}
$$

Now, if we choose the free variable $\chi_2 = 0$, then it is easy to see that $\chi_0 = \chi_1 = 0$, and we end up with the trivial solution, $x = 0$. So, instead choose $\chi_2 = 1$. (We, again, can choose any value, but it is easy to compute with 1.) Substituting this into the first two equations yields

$$
\begin{array}{ccccccc}
\chi_0 & & + & 1 & = & 0 \\
& \chi_1 & - & 2(1) & = & 0
\end{array}
$$

Solving for χ_0 and χ_1 gives us the following non-trivial solution to $Ax = 0$:

$$
x_n = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.
$$

But if $Ax_n = 0$, then $A(\beta x_n) = 0$. This means that all vectors

$$
x_s + \beta x_n = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}
$$

solve the linear system. This is the general solution that we saw before.

In this particular example, it was not necessary to exchange (pivot) rows.

Homework 9.2.5.1 Find the general solution (an expression for all solutions) for

$$
\begin{pmatrix} 2 & -2 & -4 \\ -2 & 1 & 4 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.
$$

Homework 9.2.5.2 Find the general solution (an expression for all solutions) for

$$
\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.
$$

9.3 Review of Sets

9.3.1 Definition and Notation

☞ **View at edX**

We very quickly discuss what a set is and some properties of sets. As part of discussing vector spaces, we will see lots of examples of sets and hence we keep examples down to a minimum.

Definition 9.3 *In mathematics, a set is defined as a collection of distinct objects.*

The objects that are members of a set are said to be its elements. If S is used to denote a given set and x is a member of that set, then we will use the notation $x \in S$ which is pronounced x is an element of S.

If x, y, and z are distinct objects that together are the collection that form a set, then we will often use the notation $\{x, y, z\}$ to describe that set. It is extremely important to realize that **order does not matter**: $\{x, y, z\}$ is the same set as $\{y, z, x\}$, and this is true for all ways in which you can order the objects.

A set itself is an object and hence once can have a set of sets, which has elements that are sets.

Definition 9.4 *The size of a set equals the number of distinct objects in the set.*

This size can be finite or infinite. If S denotes a set, then its size is denoted by $|S|$.

Definition 9.5 *Let S and T be sets. Then S is a subset of T if all elements of S are also elements of T. We use the notation $S \subset T$ or $T \supset S$ to indicate that S is a subset of T.*

Mathematically, we can state this as

$$
(S \subset T) \Leftrightarrow (x \in S \Rightarrow x \in T).
$$

(S is a subset of T if and only if every element in S is also an element in T.)

Definition 9.6 *Let S and T be sets. Then S is a proper subset of T if all S is a subset of T and there is an element in T that is not in S. We use the notation $S \subsetneq T$ or $T \supsetneq S$ to indicate that S is a proper subset of T.*

Some texts will use the symbol $\subset$ to mean "proper subset" and $\subseteq$ to mean "subset". Get used to it! You'll have to figure out from context what they mean.

9.3.2 Examples

☞ **View at edX**

Examples

Example 9.7 The integers $1, 2, 3$ are a collection of three objects (the given integers). The set formed by these three objects is given by $\{1, 2, 3\}$ (again, emphasizing that order doesn't matter). The size of this set is $|\{1, 2, 3,\}| = 3$.

Example 9.8 The collection of all integers is a set. It is typically denoted by $\mathbb{Z}$ and sometimes written as $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Its size is infinite: $|\mathbb{Z}| = \infty$.

Example 9.9 The collection of all real numbers is a set that we have already encountered in our course. It is denoted by $\mathbb{R}$. Its size is infinite: $|\mathbb{R}| = \infty$. We cannot enumerate it (it is uncountably infinite, which is the subject of other courses).

Example 9.10 The set of all vectors of size n whose components are real valued is denoted by $\mathbb{R}^n$.

9.3.3 Operations with Sets

☞ **View at edX**

There are three operations on sets that will be of interest:

Definition 9.11 *The **union** of two sets S and T is the set of all elements that are in S **or** in T. This union is denoted by $S \cup T$.*

Formally, we can give the union as

$$S \cup T = \{x | x \in S \lor x \in T\}$$

which is read as "The union of S and T equals the set of all elements x such that x is in S or x is in T." (The "|" (vertical bar) means "such that" and the $\lor$ is the logical "or" operator.) It can be depicted by the shaded area (blue, pink, and purple) in the following Venn diagram:
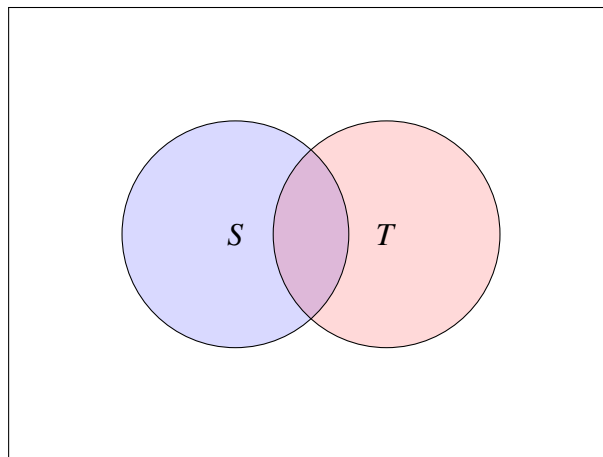
Example 9.12 Let $S = \{1, 2, 3\}$ and $T = \{2, 3, 5, 8, 9\}$. Then $S \cup T = \{1, 2, 3, 5, 8, 9\}$.
What this example shows is that the size of the union is not necessarily the sum of the sizes of the individual sets.

Definition 9.13 *The* **intersection** *of two sets S and T is the set of all elements that are in S* **and** *in T. This intersection is denoted by* $S \cap T$.

Formally, we can give the intersection as

$$S \cap T = \{x | x \in S \wedge x \in T\}$$

which is read as "The intersection of S and T equals the set of all elements x such that x is in S and x is in T." (The "|" (vertical bar) means "such that" and the $\wedge$ is the logical "and" operator.) It can be depicted by the shaded area in the following Venn diagram:
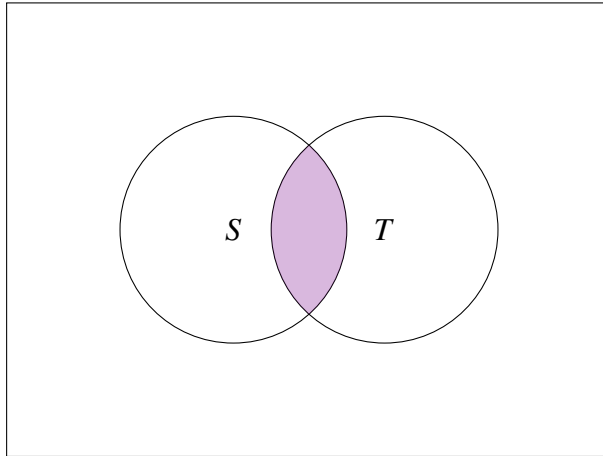
Example 9.14 Let $S = \{1,2,3\}$ and $T = \{2,3,5,8,9\}$. Then $S \cap T = \{2,3\}$.

Example 9.15 Let $S = \{1,2,3\}$ and $T = \{5,8,9\}$. Then $S \cap T = \varnothing$ ($\varnothing$ is read as "the empty set").

Definition 9.16 *The* **complement** *of set S with respect to set T is the set of all elements that are in T but are not in S. This complement is denoted by* $T \backslash S$.

Example 9.17 Let $S = \{1,2,3\}$ and $T = \{2,3,5,8,9\}$. Then $T \backslash S = \{5,8,9\}$ and $S \backslash T = \{1\}$.

Formally, we can give the complement as

$$T \backslash S = \{x | x \notin S \wedge x \in T\}$$

which is read as "The complement of S with respect to T equals the set of all elements x such that x is not in S and x is in T." (The "|" (vertical bar) means "such that", $\wedge$ is the logical "and" operator, and the $\notin$ means "is not an element in".) It can be depicted by the shaded area in the following Venn diagram:
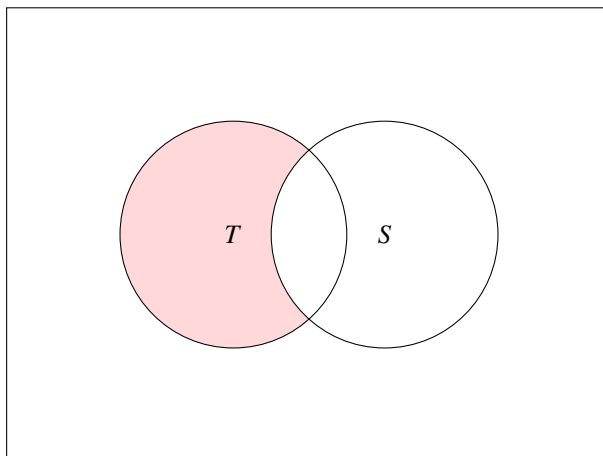
Sometimes, the notation $\bar{S}$ or S^c is used for the complement of set S. Here, the set with respect to which the complement is taken is "obvious from context".

For a single set S, the complement, $\bar{S}$ is shaded in the diagram below.

Homework 9.3.3.1 Let S and T be two sets. Then $S \subset S \cup T$.

Always/Sometimes/Never

Homework 9.3.3.2 Let S and T be two sets. Then $S \cap T \subset S$.

Always/Sometimes/Never

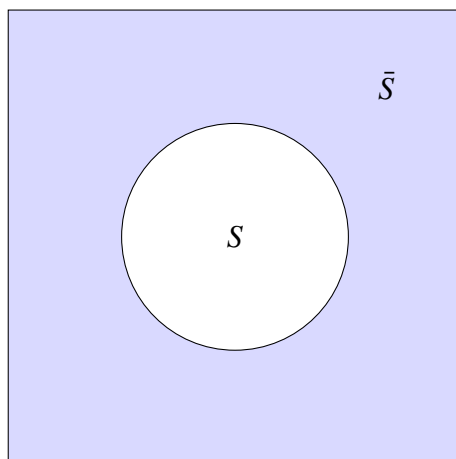## 9.4   Vector Spaces

9.4.1   What is a Vector Space?

☞ **View at edX**

For our purposes, a vector space is a subset, S, of $\mathbb{R}^n$ with the following properties:

- $0 \in S$ (the zero vector of size n is in the set S); and

- If $v, w \in S$ then $(v + w) \in S$; and

- If $\alpha \in \mathbb{R}$ and $v \in S$ then $\alpha v \in S$.

A mathematician would describe the last two properties as "S is closed under addition and scalar multiplication." All the results that we will encounter for such vector spaces carry over to the case where the components of vectors are complex valued.

Example 9.18 The set $\mathbb{R}^n$ is a vector space:

- $0 \in \mathbb{R}^n$.

- If $v, w \in \mathbb{R}^n$ then $v + w \in \mathbb{R}^n$.

- If $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then $\alpha v \in \mathbb{R}^n$.

9.4.2 Subspaces

So, the question now becomes: "What subsets of $\mathbb{R}^n$ are vector spaces?" We will call such sets **subspaces** of $\mathbb{R}^n$.

Homework 9.4.2.1 Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

1. The plane of vectors $x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}$ such that $\chi_0 = 0$. In other words, the set of all vectors

$$\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} 0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \right\}.$$

2. Similarly, the plane of vectors x with $\chi_0 = 1$: $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} 1 \\ \chi_1 \\ \chi_2 \end{pmatrix} \right\}.$

3. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \wedge \chi_0 \chi_1 = 0 \right\}$. (Recall, $\wedge$ is the logical "and" operator.)

4. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \beta_0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ where } \beta_0, \beta_1 \in \mathbb{R} \right\}.$

5. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \wedge \chi_0 - \chi_1 + 3\chi_2 = 0 \right\}.$

Homework 9.4.2.2 The empty set, $\varnothing$, is a subspace of $\mathbb{R}^n$.

True/False

Homework 9.4.2.3 The set $\{0\}$ where 0 is a vector of size n is a subspace of $\mathbb{R}^n$.

True/False

Homework 9.4.2.4 The set $S \subset \mathbb{R}^n$ described by

$$\{x \mid \|x\|_2 < 1\}.$$

is a subspace of $\mathbb{R}^n$. (Recall that $\|x\|_2$ is the Euclidean length of vector x so this describes all elements with length less than or equal to one.)

True/False

Homework 9.4.2.5 The set $S \subset \mathbb{R}^n$ described by

$$\left\{ \left. \begin{pmatrix} \nu_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \nu_0 \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^n$.

True/False

Homework 9.4.2.6 The set $S \subset \mathbb{R}^n$ described by

$$\left\{ \nu e_j \mid \nu \in \mathbb{R} \right\},$$

where e_j is a unit basis vector, is a subspace.

True/False
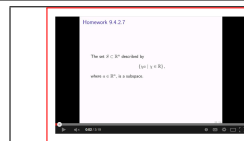
Homework 9.4.2.7 The set $S \subset \mathbb{R}^n$ described by

$$\left\{ \chi a \mid \chi \in \mathbb{R} \right\},$$

where $a \in \mathbb{R}^n$, is a subspace.

True/False

Homework 9.4.2.8 The set $S \subset \mathbb{R}^n$ described by

$$\left\{ \chi_0 a_0 + \chi_1 a_1 \mid \chi_0, \chi_1 \in \mathbb{R} \right\},$$

where $a_0, a_1 \in \mathbb{R}^n$, is a subspace.

True/False

Homework 9.4.2.9 The set $S \subset \mathbb{R}^m$ described by

$$\left\{ \left. \left(a_0 \mid a_1 \right) \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right| \chi_0, \chi_1 \in \mathbb{R} \right\},$$

where $a_0, a_1 \in \mathbb{R}^m$, is a subspace.

True/False

Homework 9.4.2.10 The set $S \subset \mathbb{R}^m$ described by

$$\left\{ Ax \mid x \in \mathbb{R}^2 \right\},$$

where $A \in \mathbb{R}^{m \times 2}$, is a subspace.

True/False

9.4.3 The Column Space

Homework 9.4.3.1 The set $S \subset \mathbb{R}^m$ described by

$$\{Ax \mid x \in \mathbb{R}^n\},$$

where $A \in \mathbb{R}^{m \times n}$, is a subspace.

True/False

This last exercise very precisely answers the question of when a linear system of equation, expressed as the matrix equation $Ax = b$, has a solution: it has a solution only if b is an element of the space S in this last exercise.

Definition 9.19 *Let $A \in \mathbb{R}^{m \times n}$. Then the* **column space** *of A equals the set*

$$\{Ax \mid x \in \mathbb{R}^n\}.$$

It is denoted by $\mathcal{C}(A)$.

The name "column space" comes from the observation (which we have made many times by now) that

$$Ax = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}.$$

Thus $\mathcal{C}(A)$ equals the set of all linear combinations of the columns of matrix A.

Theorem 9.20 *The column space of $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^m$.*

Proof: The last exercise proved this.

Theorem 9.21 *Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Then $Ax = b$ has a solution if and only if $b \in \mathcal{C}(A)$.*

Proof: Recall that to prove an "if and only if" statement $P \Leftrightarrow Q$, you may want to instead separately prove $P \Rightarrow Q$ and $P \Leftarrow Q$.

($\Rightarrow$) Assume that $Ax = b$. Then $b \in \{Ax \mid x \in \mathbb{R}^n\}$. Hence b is in the column space of A.

($\Leftarrow$) Assume that b is in the column space of A. Then $b \in \{Ax \mid x \in \mathbb{R}^n\}$. But this means there exists a vector x such that $Ax = b$.

Homework 9.4.3.2 Match the matrices on the left to the column space on the right. (You should be able to do this "by examination.")

1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

a. $\mathbb{R}^2$.

3. $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$

b. $\left\{ \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \middle| \chi_0 = 0 \vee \chi_1 = 0 \right\}$

4. $\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$

c. $\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

5. $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

d. $\left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

6. $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

e. $\left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

7. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

8. $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$

f. $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

9. $\begin{pmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \end{pmatrix}$

(Recall that $\vee$ is the logical "or" operator.)

Homework 9.4.3.3 Which of the following matrices have a FINITE number of elements in their column space? (Mark all that apply.)

1. The identity matrix.

2. The zero matrix.

3. All matrices.

4. None of the above.

9.4.4 The Null Space

☞ **View at edX**

Recall:

- We are interested in the solutions of $Ax = b$.

- We have already seen that if $Ax_s = b$ and $Ax_n = 0$ then $x_s + x_n$ is also a solution:

$$A(x_s + x_n) = b.$$

Definition 9.22 *Let $A \in \mathbb{R}^{m \times n}$. Then the set of all vectors $x \in \mathbb{R}^n$ that have the property that $Ax = 0$ is called the null space of A and is denoted by*

$$\mathcal{N}(A) = \{x | Ax = 0\}.$$

Homework 9.4.4.1 Let $A \in \mathbb{R}^{m \times n}$. The null space of A, $\mathcal{N}(A)$, is a subspace

True/False

Homework 9.4.4.2 For each of the matrices on the left match the set of vectors on the right that describes its null space. (You should be able to do this "by examination.")

a. $\mathbb{R}^2$.

1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

b. $\left\{ \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \middle| \chi_0 = 0 \vee \chi_1 = 0 \right\}$

2. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c. $\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

3. $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$

d. $\varnothing$

4. $\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$

e. $\left\{ \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

5. $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

f. $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

6. $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

g. $\left\{ \begin{pmatrix} 0 \end{pmatrix} \right\}$

7. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

h. $\left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

8. $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$

i. $\left\{ \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

(Recall that $\vee$ is the logical "or" operator.)

9.5 Span, Linear Independence, and Bases

9.5.1 Span

☛ **View at edX**

What is important about vector (sub)spaces is that if you have one or more vectors in that space, then it is possible to generate other vectors in the subspace by taking linear combinations of the original known vectors.

Example 9.23

$$\left\{ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \alpha_0, \alpha_1 \in \mathbb{R} \right\}$$

is the set of all linear combinations of the unit basis vectors $e_0, e_1 \in \mathbb{R}^2$. Notice that all of $\mathbb{R}^2$ (an uncountable infinite set) can be described with just these two vectors.

We have already seen that, given a set of vectors, the set of all linear combinations of those vectors is a subspace. We now give a name to such a set of linear combinations.

Definition 9.24 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$. Then the span of these vectors, $Span\{v_0, v_1, \cdots, v_{n-1}\}$, is said to be the set of all vectors that are a linear combination of the given set of vectors.*

Example 9.25

$$\text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \mathbb{R}^2.$$

Example 9.26 Consider the equation $\chi_0 + 2\chi_1 - \chi_2 = 0$. It defines a subspace. In particular, that subspace is the null space of the matrix $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$. We know how to find two vectors in that nullspace:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \end{array} \right)$$

The box identifies the pivot. Hence, the free variables are χ_1 and χ_2. We first set $\chi_1 = 1$ and $\chi_2 = 0$ and solve for χ_0. Then we set $\chi_1 = 0$ and $\chi_2 = 1$ and again solve for χ_0. This gives us the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

We know that any linear combination of these vectors also satisfies the equation (is also in the null space). Hence, we know that any vector in

$$\text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right)$$

is also in the null space of the matrix. Thus, any vector in that set satisfies the equation given at the start of this example.

We will later see that the vectors in this last example "span" the entire null space for the given matrix. But we are not quite ready to claim that.

We have learned three things in this course that relate to this discussion:

- Given a set of vectors $\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathbb{R}^n$, we can create a matrix that has those vectors as its columns:

$$V = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_{n-1} \end{array} \right).$$

- Given a matrix $V \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$,

$$Vx = \chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1}.$$

In other words, Vx takes a linear combination of the columns of V.

- The column space of V, $C(V)$, is the set (subspace) of all linear combinations of the columns of V:

$$C(V) = \{Vx \,|\, x \in \mathbb{R}^n\} = \{\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1} \,|\, \chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathbb{R}\}.$$

We conclude that

> If $V = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_{n-1} \end{array} \right)$, then $\mathrm{Span}(v_0, v_1, \ldots, v_{n-1}) = C(V)$.

Definition 9.27 *A spanning set of a subspace S is a set of vectors $\{v_0, v_1, \ldots, v_{n-1}\}$ such that $\mathrm{Span}(\{v_0, v_1, \ldots, v_{n-1}\}) = S$.*

9.5.2 Linear Independence

☞ **View at edX**

Example 9.28 We show that $\mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right) = \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$. One can either simply recognize that both sets equal all of $\mathbb{R}^2$, or one can reason it by realizing that in order to show that sets S and T are equal one can just show that both $S \subset T$ and $T \subset S$:

- $S \subset T$: Let $x \in \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$ Then there exist α_0 and α_1 such that $x = \alpha_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This in turn means

  that $x = \alpha_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (0) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence

  $$x \in \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right).$$

- $T \subset S$: Let $x \in \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$. Then there exist α_0, α_1, and α_2 such that $x = \alpha_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} +$

  $\alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. But $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence

  $$x = \alpha_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = (\alpha_0 + \alpha_2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (\alpha_1 + \alpha_2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

  Therefore $x \in \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$.

Homework 9.5.2.1

$$
\text{Span} \left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right) = \text{Span} \left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\} \right)
$$

True/False

You might be thinking that needing fewer vectors to describe a subspace is better than having more, and we'd agree with you!

In both examples and in the homework, the set on the right of the equality sign identifies three vectors to identify the subspace rather than the two required for the equivalent set to its left. The issue is that at least one (indeed all) of the vectors can be written as linear combinations of the other two. Focusing on the exercise, notice that

$$
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
$$

Thus, any linear combination

$$
\alpha_0 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}
$$

can also be generated with only the first two vectors:

$$
\alpha_0 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = (\alpha_0 + \alpha_2) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (\alpha_0 + 2\alpha_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

We now introduce the concept of linear (in)dependence to cleanly express when it is the case that a set of vectors has elements that are redundant in this sense.

Definition 9.29 *Let* $\{v_0, \ldots, v_{n-1}\} \subset \mathbb{R}^m$. *Then this set of vectors is said to be linearly independent if* $\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1} = 0$ *implies that* $\chi_0 = \cdots = \chi_{n-1} = 0$. *A set of vectors that is not linearly independent is said to be linearly dependent.*

Homework 9.5.2.2 Let the set of vectors $\{a_0, a_1, \ldots, a_{n-1}\} \subset \mathbb{R}^m$ be linearly dependent. Then at least one of these vectors can be written as a linear combination of the others.

True/False

This last exercise motivates the term *linearly independent* in the definition: none of the vectors can be written as a linear combination of the other vectors.

Example 9.30 The set of vectors

$$
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}
$$

is linearly dependent:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Theorem 9.31 *Let* $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^m$ *and let* $A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right)$. *Then the vectors* $\{a_0, \ldots, a_{n-1}\}$ *are linearly independent if and only if* $\mathcal{N}(A) = \{0\}$.

Proof:

($\Rightarrow$) Assume $\{a_0, \ldots, a_{n-1}\}$ are linearly independent. We need to show that $\mathcal{N}(A) = \{0\}$. Assume $x \in \mathcal{N}(A)$. Then $Ax = 0$ implies that

$$
\begin{aligned}
0 &= Ax = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right) \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \\
&= \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}
\end{aligned}
$$

and hence $\chi_0 = \cdots = \chi_{n-1} = 0$. Hence $x = 0$.

($\Leftarrow$) Notice that we are trying to prove $P \Leftarrow Q$, where P represents "the vectors $\{a_0, \ldots, a_{n-1}\}$ are linearly independent" and Q represents "$\mathcal{N}(A) = \{0\}$". It suffices to prove the **contrapositive**: $\neg P \Rightarrow \neg Q$. (Note that $\neg$ means "not") Assume that $\{a_0, \ldots, a_{n-1}\}$ are *not* linearly independent. Then there exist $\{\chi_0, \cdots, \chi_{n-1}\}$ with at least one $\chi_j \neq 0$ such that $\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} = 0$. Let $x = (\chi_0, \ldots, \chi_{n-1})^T$. Then $Ax = 0$ which means $x \in \mathcal{N}(A)$ and hence $\mathcal{N}(A) \neq \{0\}$.

Example 9.32 In the last example, we could have taken the three vectors to be the columns of a 3×3 matrix A and checked if $Ax = 0$ has a solution:

$$
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Because there is a non-trivial solution to $Ax = 0$, the nullspace of A has more than just the zero vector in it, and the columns of A are linearly dependent.

Example 9.33 The columns of an identity matrix $I \in \mathbb{R}^{n \times n}$ form a linearly independent set of vectors.

Proof: Since I has an inverse (I itself) we know that $\mathcal{N}(I) = \{0\}$. Thus, by Theorem 9.31, the columns of I are linearly independent.

Example 9.34 The columns of $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ are linearly independent. If we consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and simply solve this, we find that $\chi_0 = 0/1 = 0$, $\chi_1 = (0 - 2\chi_0)/(-1) = 0$, and $\chi_2 = (0 - \chi_0 - 2\chi_1)/(3) = 0$. Hence, $\mathcal{N}(L) = \{0\}$ (the zero vector) and we conclude, by Theorem 9.31, that the columns of L are linearly independent.

The last example motivates the following theorem:

Theorem 9.35 *Let $L \in \mathbb{R}^{n \times n}$ be a lower triangular matrix with nonzeroes on its diagonal. Then its columns are linearly independent.*

Proof: Let L be as indicated and consider $Lx = 0$. If one solves this via whatever method one pleases, the solution $x = 0$ will emerge as the only solution. Thus $\mathcal{N}(L) = \{0\}$ and by Theorem 9.31, the columns of L are linearly independent.

Homework 9.5.2.3 Let $U \in \mathbb{R}^{n \times n}$ be an upper triangular matrix with nonzeroes on its diagonal. Then its columns are linearly independent.   Always/Sometimes/Never

Homework 9.5.2.4 Let $L \in \mathbb{R}^{n \times n}$ be a lower triangular matrix with nonzeroes on its diagonal. Then its rows are linearly independent. (Hint: How do the rows of L relate to the columns of L^T?)
                                                                                    Always/Sometimes/Never

Example 9.36 The columns of $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \\ \hline -1 & 0 & -2 \end{pmatrix}$ are linearly independent. If we consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 3 \\ \hline -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and simply solve this, we find that $\chi_0 = 0/1 = 0$, $\chi_1 = (0 - 2\chi_0)/(-1) = 0$, $\chi_2 = (0 - \chi_0 - 2\chi_1)/(3) = 0$. Hence, $\mathcal{N}(L) = \{0\}$ (the zero vector) and we conclude, by Theorem 9.31, that the columns of L are linearly independent.

Next, we observe that if one has a set of more than m vectors in $\mathbb{R}^m$, then they must be linearly dependent:

Theorem 9.37 *Let $\{a_0, a_1, \ldots, a_{n-1}\} \in \mathbb{R}^m$ and $n > m$. Then these vectors are linearly dependent.*

Proof: Consider the matrix $A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right)$. If one applies the Gauss-Jordan method to this matrix in order to get it to upper triangular form, at most m columns with pivots will be encountered. The other $n - m$ columns correspond to free variables, which allow us to construct nonzero vectors x so that $Ax = 0$.

The observations in this unit allows us to add to our conditions related to the invertibility of matrix A:

The following statements are equivalent statements about $A \in \mathbb{R}^{n \times n}$:

- A is nonsingular.

- A is invertible.

- A^{-1} exists.

- $AA^{-1} = A^{-1}A = I$.

- A represents a linear transformation that is a bijection.

- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.

- $Ax = 0$ implies that $x = 0$.

- $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.

- The determinant of A is nonzero: $\det(A) \neq 0$.

- LU with partial pivoting does not break down.

- $\mathcal{C}(A) = \mathbb{R}^n$.

- A has linearly independent columns.

- $\mathcal{N}(A) = \{0\}$.

9.5.3 Bases for Subspaces

☞ **View at edX**

In the last unit, we started with an example and then an exercise that showed that if we had three vectors and one of the three vectors could be written as a linear combination of the other two, then the span of the three vectors was equal to the span of the other two vectors.

It turns out that this can be generalized:

Definition 9.38 *Let S be a subspace of $\mathbb{R}^m$. Then the set $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ is said to be a basis for S if (1) $\{v_0, v_1, \cdots, v_{n-1}\}$ are linearly independent and (2) $\text{Span}\{v_0, v_1, \cdots, v_{n-1}\} = S$.*

Homework 9.5.3.1 The vectors $\{e_0, e_1, \ldots, e_{n-1}\} \subset \mathbb{R}^n$ are a basis for $\mathbb{R}^n$.

True/False

Example 9.39 Let $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$ and let $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$ be invertible. Then $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Note: The fact that A is invertible means there exists A^{-1} such that $A^{-1}A = I$. Since $Ax = 0$ means $x = A^{-1}Ax = A^{-1}0 = 0$, the columns of A are linearly independent. Also, given any vector $y \in \mathbb{R}^n$, there exists a vector $x \in \mathbb{R}^n$ such that $Ax = y$ (namely $x = A^{-1}y$). Letting $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$ we find that $y = \chi_0 a_0 + \cdots + \chi_{n-1} a_{n-1}$ and hence every vector in $\mathbb{R}^n$ is a linear combination of the set $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$.

Lemma 9.40 *Let $S \subset \mathbb{R}^m$. Then S contains at most m linearly independent vectors.*

Proof: Proof by contradiction. We will assume that S contains more than m linearly independent vectors and show that this leads to a contradiction.

Since S contains more than m linearly independent vectors, it contains at least $m + 1$ linearly independent vectors. Let us label $m + 1$ such vectors $v_0, v_1, \ldots, v_{m-1}, v_m$. Let $V = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_m \end{array} \right)$. This matrix is $m \times (m+1)$ and hence there exists a nontrivial x_n such that $Vx_n = 0$. (This is an equation with m equations and $m + 1$ unknowns.) Thus, the vectors $\{v_0, v_1, \cdots, v_m\}$ are linearly dependent, which is a contradiction.

Theorem 9.41 *Let S be a nontrivial subspace of $\mathbb{R}^m$. (In other words, $S \neq \{0\}$.) Then there exists a basis $\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathbb{R}^m$ such that $Span(v_0, v_1, \ldots, v_{n-1}) = S$.*

Proof: Notice that we have already established that $m < n$. We will construct the vectors. Let S be a nontrivial subspace. Then S contains at least one nonzero vector. Let v_0 equal such a vector. Now, either $Span(v_0) = S$ in which case we are done or $S \backslash Span(v_0)$ is not empty, in which case we can pick some vector in $S \backslash Span(v_0)$ as v_1. Next, either $Span(v_0, v_1) = S$ in which case we are done or $S \backslash Span(v_0, v_1)$ is not empty, in which case we pick some vector in $S \backslash Span(v_0, v_1)$ as v_2. This process continues until we have a basis for S. It can be easily shown that the vectors are all linearly independent.

9.5.4 The Dimension of a Subspace

☞ **View at edX**

We have established that every nontrivial subspace of $\mathbb{R}^m$ has a basis with n vectors. This basis is not unique. After all, we can simply multiply all the vectors in the basis by a nonzero constant and contruct a new basis. What we'll establish now is that the number of vectors in a basis for a given subspace is always the same. This number then becomes the dimension of the subspace.

Theorem 9.42 *Let S be a subspace of $\mathbb{R}^m$ and let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ and $\{w_0, w_1, \cdots, w_{k-1}\} \subset \mathbb{R}^m$ both be bases for S. Then $k = n$. In other words, the number of vectors in a basis is unique.*

Proof: Proof by contradiction. Without loss of generality, let us assume that $k > n$. (Otherwise, we can switch the roles of the two sets.) Let $V = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array} \right)$ and $W = \left(\begin{array}{c|c|c} w_0 & \cdots & w_{k-1} \end{array} \right)$. Let x_j have the property that $w_j = V x_j$. (We know such a vector x_j exists because V spans $\mathbf{V}$ and $w_j \in \mathbf{V}$.) Then $W = VX$, where $X = \left(\begin{array}{c|c|c} x_0 & \cdots & x_{k-1} \end{array} \right)$. Now, $X \in \mathbb{R}^{n \times k}$ and recall that $k > n$. This means that $\mathcal{N}(X)$ contains nonzero vectors (why?). Let $y \in \mathcal{N}(X)$. Then $Wy = VXy = V(Xy) = V(0) = 0$, which contradicts the fact that $\{w_0, w_1, \cdots, w_{k-1}\}$ are linearly independent, and hence this set cannot be a basis for $\mathbf{V}$.

Definition 9.43 *The dimension of a subspace S equals the number of vectors in a basis for that subspace.*

A basis for a subspace S can be derived from a spanning set of a subspace S by, one-to-one, removing vectors from the set that are dependent on other remaining vectors until the remaining set of vectors is linearly independent , as a consequence of the following observation:

Definition 9.44 *Let $A \in \mathbb{R}^{m \times n}$. The rank of A equals the number of vectors in a basis for the column space of A. We will let* rank(A) *denote that rank.*

Theorem 9.45 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be a spanning set for subspace S and assume that v_i equals a linear combination of the other vectors. Then $\{v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n-1}\}$ is a spanning set of S.*

Similarly, a set of linearly independent vectors that are in a subspace S can be "built up" to be a basis by successively adding vectors that are in S to the set while maintaining that the vectors in the set remain linearly independent until the resulting is a basis for S.

Theorem 9.46 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be linearly independent and assume that $\{v_0, v_1, \cdots, v_{n-1}\} \subset S$ where S is a subspace. Then this set of vectors is either a spanning set for S or there exists $w \in S$ such that $\{v_0, v_1, \cdots, v_{n-1}, w\}$ are linearly independent.*

We can add some more conditions regarding the invertibility of matrix A:

The following statements are equivalent statements about $A \in \mathbb{R}^{n \times n}$:

- A is nonsingular.

- A is invertible.

- A^{-1} exists.

- $AA^{-1} = A^{-1}A = I$.

- A represents a linear transformation that is a bijection.

- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.

- $Ax = 0$ implies that $x = 0$.

- $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.

- The determinant of A is nonzero: $\det(A) \neq 0$.

- LU with partial pivoting does not break down.

- $\mathcal{C}(A) = \mathbb{R}^n$.

- A has linearly independent columns.

- $\mathcal{N}(A) = \{0\}$.

- rank$(A) = n$.

9.6   Enrichment

9.6.1   Typesetting algorithms with the FLAME notation

☛ **View at edX**

9.7   Wrap Up

9.7.1   Homework

No additional homework this week.

9.7.2   Summary

Solution(s) to linear systems

Whether a linear system of equations $Ax = b$ has a unique solution, no solution, or multiple solutions can be determined by writing the system as an appended system

$$\left(\begin{array}{c|c} A & b \end{array} \right)$$

and transforming this appended system to row echelon form, swapping rows if necessary.

When A is square, conditions for the solution to be unique were discussed in Weeks 6-8.

Examples of when it has a unique solution, no solution, or multiple solutions when $m \neq n$ were given in this week, but this will become more clear in Week 10. Therefore, we won't summarize it here.

Sets

Definition 9.47 *In mathematics, a set is defined as a collection of distinct objects.*

- The objects that are members of a set are said to be its elements.

- The notation $x \in S$ is used to indicate that x is an element in set S.

Definition 9.48 *The size of a set equals the number of distinct objects in the set. It is denoted by $|S|$.*

Definition 9.49 *Let S and T be sets. Then S is a subset of T if all elements of S are also elements of T. We use the notation $S \subset T$ to indicate that S is a subset of T:*

$$(S \subset T) \Leftrightarrow (x \in S \Rightarrow x \in T).$$

Definition 9.50 *The* **union** *of two sets S and T is the set of all elements that are in S* **or** *in T. This union is denoted by $S \cup T$:*

$$S \cup T = \{x | x \in S \lor x \in T.\}$$

Definition 9.51 *The* **intersection** *of two sets S and T is the set of all elements that are in S* **and** *in T. This intersection is denoted by $S \cap T$:*

$$S \cap T = \{x | x \in S \land x \in T.\}$$

Definition 9.52 *The* **complement** *of set S with respect to set T is the set of all elements that are in T but are not in S. This complement is denoted by $T \backslash S$:*

$$T \backslash S = \{x | x \notin S \land x \in T\}$$

Vector spaces

For our purposes, a vector space is a subset, S, of $\mathbb{R}^n$ with the following properties:

- $0 \in S$ (the zero vector of size n is in the set S); and

- If $v, w \in S$ then $(v + w) \in S$; and

- If $\alpha \in \mathbb{R}$ and $v \in S$ then $\alpha v \in S$.

Definition 9.53 *A subset of $\mathbb{R}^n$ is said to be a subspace of $\mathbb{R}^n$ is it a vector space.*

Definition 9.54 *Let $A \in \mathbb{R}^{m \times n}$. Then the* **column space** *of A equals the set*

$$\{ Ax \mid x \in \mathbb{R}^n \}.$$

It is denoted by $C(A)$.

The name "column space" comes from the observation (which we have made many times by now) that

$$Ax = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}.$$

Thus $C(A)$ equals the set of all linear combinations of the columns of matrix A.

Theorem 9.55 *The column space of $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^m$.*

Theorem 9.56 *Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Then $Ax = b$ has a solution if and only if $b \in C(A)$.*

Definition 9.57 *Let $A \in \mathbb{R}^{m \times n}$. Then the set of all vectors $x \in \mathbb{R}^n$ that have the property that $Ax = 0$ is called the null space of A and is denoted by*

$$\mathcal{N}(A) = \{ x | Ax = 0 \}.$$

Span, Linear Dependence, Bases

Definition 9.58 *Let $\{ v_0, v_1, \cdots, v_{n-1} \} \subset \mathbb{R}^m$. Then the span of these vectors, $Span\{ v_0, v_1, \cdots, v_{n-1} \}$, is said to be the set of all vectors that are a linear combination of the given set of vectors.*

If $V = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_{n-1} \end{array} \right)$, then $Span(v_0, v_1, \ldots, v_{n-1}) = C(V)$.

Definition 9.59 *A spanning set of a subspace S is a set of vectors $\{ v_0, v_1, \ldots, v_{n-1} \}$ such that $Span(\{ v_0, v_1, \ldots, v_{n-1} \}) = S$.*

Definition 9.60 *Let $\{ v_0, \ldots, v_{n-1} \} \subset \mathbb{R}^m$. Then this set of vectors is said to be linearly independent if $\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1} = 0$ implies that $\chi_0 = \cdots = \chi_{n-1} = 0$. A set of vectors that is not linearly independent is said to be linearly dependent.*

Theorem 9.61 *Let the set of vectors $\{ a_0, a_1, \ldots, a_{n-1} \} \subset \mathbb{R}^m$ be linearly dependent. Then at least one of these vectors can be written as a linear combination of the others.*

This last theorem motivates the term *linearly independent* in the definition: none of the vectors can be written as a linear combination of the other vectors.

Theorem 9.62 *Let $\{ a_0, \ldots, a_{n-1} \} \subset \mathbb{R}^m$ and let $A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right)$. Then the vectors $\{ a_0, \ldots, a_{n-1} \}$ are linearly independent if and only if $\mathcal{N}(A) = \{ 0 \}$.*

Theorem 9.63 *Let $\{a_0, a_1, \ldots, a_{n-1}\} \in \mathbb{R}^m$ and $n > m$. Then these vectors are linearly dependent.*

Definition 9.64 *Let S be a subspace of $\mathbb{R}^m$. Then the set $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ is said to be a basis for S if (1) $\{v_0, v_1, \cdots, v_{n-1}\}$ are linearly independent and (2) $\text{Span}\{v_0, v_1, \cdots, v_{n-1}\} = S$.*

Theorem 9.65 *Let S be a subspace of $\mathbb{R}^m$ and let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ and $\{w_0, w_1, \cdots, w_{k-1}\} \subset \mathbb{R}^m$ both be bases for S. Then $k = n$. In other words, the number of vectors in a basis is unique.*

Definition 9.66 *The dimension of a subspace S equals the number of vectors in a basis for that subspace.*

Definition 9.67 *Let $A \in \mathbb{R}^{m \times n}$. The rank of A equals the number of vectors in a basis for the column space of A. We will let* rank(A) *denote that rank.*

Theorem 9.68 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be a spanning set for subspace S and assume that v_i equals a linear combination of the other vectors. Then $\{v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n-1}\}$ is a spanning set of S.*

Theorem 9.69 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be linearly independent and assume that $\{v_0, v_1, \cdots, v_{n-1}\} \subset S$ where S is a subspace. Then this set of vectors is either a spanning set for S or there exists $w \in S$ such that $\{v_0, v_1, \cdots, v_{n-1}, w\}$ are linearly independent.*

The following statements are equivalent statements about $A \in \mathbb{R}^{n \times n}$:

- A is nonsingular.

- A is invertible.

- A^{-1} exists.

- $AA^{-1} = A^{-1}A = I$.

- A represents a linear transformation that is a bijection.

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- $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.

- The determinant of A is nonzero: $\det(A) \neq 0$.

- LU with partial pivoting does not break down.

- $\mathcal{C}(A) = \mathbb{R}^n$.

- A has linearly independent columns.

- $\mathcal{N}(A) = \{0\}$.

- rank$(A) = n$.