Deriving the LU Factorization Algorithm (Unblocked Variant 5)

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Outline

Operation: LU Factorization (LU)

Goal oriented programming

The "worksheet"

Step 1: Precondition and postcondition

Step 2: Deriving the invariants

Step 3: Loop guard

Step 4: Initialization

Step 5: Moving through the matrices

Step 6: State before update statements

Step 7: State after update statements

Step 8: The update statements

The algorithm

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The algorithm

Given $A \in \mathbb{R}^{n \times n}$ compute unit lower triangular matrix L and upper triangular matrix U such that $A = LU$.

Algorithm in FLAME Notation

Algorithm: $A := \text{LU\_UNB\_VAR5}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

  where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

  Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

  where α_{11} is 1×1

$a_{21} := a_{21}/\alpha_{11}$

$A_{22} := A_{22} - a_{21}a_{12}^T$

  Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Proving an algorithm correct

A loop invariant is the state of (in this case) matrix A before and after each iteration of the loop.

Step	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	**Partition** $A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$, $L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right)$, $U \rightarrow \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right)$ **where** A_{TL}, L_{TL}, and U_{TL} are 0×0
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge (m(A_{TL}) < m(A)) \right\}$
5a	**Repartition** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$ **where** α_{11}, 1, and υ_{11} are 1×1
6	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c\|c\|c} L \backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \widehat{a}_{21} - L_{20} u_{01} & \widehat{A}_{22} - L_{20} U_{20} \end{array} \right) \wedge \begin{array}{c\|c\|c} L_{00} U_{00} = \widehat{A}_{00} & L_{00} u_{01} = \widehat{a}_{01} & L_{00} U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ \hline L_{20} U_{00} = \widehat{A}_{20} & & \end{array} \right\}$
8	$a_{21} := a_{21} / \alpha_{11}$ $A_{22} := A_{22} - a_{21} a_{12}^T$
5b	**Continue with** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$
7	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c\|c\|c} L \backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20} U_{20} - l_{21} u_{12}^T \end{array} \right) \wedge \begin{array}{c\|c\|c} L_{00} U_{00} = \widehat{A}_{00} & L_{00} u_{01} = \widehat{a}_{01} & L_{00} U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20} U_{00} = \widehat{A}_{20} & L_{20} u_{01} + l_{21} \upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
	endwhile
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg (m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

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The algorithm

"The only effective way to raise the confidence level of a program significantly is to give a convincing proof of its correctness. But one should not first make the program and then prove its correctness, because then the requirement of providing the proof would only increase the poor programmers burden. On the contrary: the programmer should let correctness proof and program grow hand in hand."

– Dijkstra

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The algorithm

The “worksheet” structures the derivation of the algorithm hand-in-hand with the proof of its correctness.

Step	Annotated Algorithm: $A := \mathrm{LU}(A)$
1a	
4	
2	
3	**while** **do**
2,3	
5a	
6	
8	
5b	
7	
2	
	endwhile
2,3	
1b	

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The algorithm

$$A := \mathrm{LU}(A)$$

- Precondition: $A = \widehat{A}$
- Postcondition: $A = L \backslash U \wedge LU = \widehat{A}$

Step	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	
2	
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	
5a	
6	
8	
5b	
7	
2	
	endwhile
2,3	
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

ep	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
a	$\left\{ A = \widehat{A} \right\}$
4	
2	
3	**while** **do**

$\vdots$

	endwhile
13	
b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

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The algorithm

Partitioned Matrix Expression

Consider the postcondition: $A = L\backslash U \wedge LU = \widehat{A}$. Substitute partitioned matrices and/or vectors

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right), L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right), U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$$

to yield

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right)$$

$$\text{where} \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) = \left(\begin{array}{c|c} \widehat{A}_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right)$$

Linear algebra yields the Partitioned Matrix Expression (PME):

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

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Determining loop-invariants

Start with the PME:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right)$$

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The PME captures *all* computation that must be performed in order to compute the *final* result, in terms of the submatrices/vectors that are encountered.

A loop invariant indicates *some* of this computation has been completed. For example

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \require{cancel}\cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$$

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A loop invariant indicates *some* of this computation has been completed. For example

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$$

PME: $$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

Invariant	
1	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline \cancel{L_{BL}U_{TL} = \widehat{A}_{BL}} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
2	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
3	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \end{array}$

PME: $$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

<table>
<tr><th>Invariant</th><th></th></tr>
<tr><td>1</td><td>$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$</td></tr>
<tr><td>2</td><td>$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$</td></tr>
<tr><td>3</td><td>$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right)$ $L_{TL}U_{TL} = \widehat{A}_{TL} \mid L_{TL}U_{TR} = \widehat{A}_{TR}$</td></tr>
</table>

PME: $$\begin{pmatrix} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR}\end{pmatrix} = \begin{pmatrix} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR}\end{pmatrix} \wedge \begin{array}{cc} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}\end{array}$$

Invariant	
1	$\begin{pmatrix} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR}\end{pmatrix} = \begin{pmatrix} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR}\end{pmatrix} \wedge \begin{array}{cc} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline \cancel{L_{BL}U_{TL} = \widehat{A}_{BL}} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}}\end{array}$
2	$\begin{pmatrix} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR}\end{pmatrix} = \begin{pmatrix} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline L_{BL} & \widehat{A}_{BR}\end{pmatrix} \wedge \begin{array}{cc} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}}\end{array}$
3	$\begin{pmatrix} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR}\end{pmatrix} = \begin{pmatrix} L\backslash U_{TL} & U_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR}\end{pmatrix}$ $L_{TL}U_{TL} = \widehat{A}_{TL}$ $L_{TL}U_{TR} = \widehat{A}_{TR}$

PME: $$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

Invariant	
1	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline \cancel{L_{BL}U_{TL} = \widehat{A}_{BL}} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
2	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & \widehat{A}_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & \cancel{L_{TL}U_{TR} = \widehat{A}_{TR}} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
3	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right)$ $L_{TL}U_{TL} = \widehat{A}_{TL}$ | $L_{TL}U_{TR} = \widehat{A}_{TR}$

PME:
$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right)$$
$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

Invariant	
4	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
5	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$

PME:
$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & L\backslash U_{BR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR} \end{array}$$

Invariant	
4	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$
5	$\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$

We choose Invariant 5:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \widehat{A}_{BR}} \end{array}$$

<table>
<tr><th>Step</th><th>Annotated Algorithm: $A := \mathrm{LU}(A)$</th></tr>
<tr><td>1a</td><td>$\left\{ A = \widehat{A} \right\}$</td></tr>
<tr><td>4</td><td></td></tr>
<tr><td>2</td><td>$\left\{ \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$</td></tr>
<tr><td>3</td><td>while $m(A_{TL}) < m(A)$ do</td></tr>
<tr><td>2,3</td><td>$\left\{ \left(\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \right\}$</td></tr>
<tr><td>5a</td><td></td></tr>
<tr><td>6</td><td></td></tr>
<tr><td>8</td><td></td></tr>
<tr><td>5b</td><td></td></tr>
<tr><td>7</td><td></td></tr>
<tr><td>2</td><td>$\left\{ \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$</td></tr>
<tr><td></td><td>endwhile</td></tr>
<tr><td>2,3</td><td>$\left\{ \left(\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \right\}$</td></tr>
<tr><td>1b</td><td>$\left\{ A = L\backslash U \wedge LU = \widehat{A} \right\}$</td></tr>
</table>

ep	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
a	$\left\{ A = \widehat{A} \right\}$
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \right.$

$$\vdots$$

2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
	endwhile
3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \right.$
b	$\left\{ A = L\backslash U \wedge LU = \widehat{A} \right\}$

Operation: LU Factorization (LU)

Goal oriented programming

The "worksheet"

Step 1: Precondition and postcondition

Step 2: Deriving the invariants

Step 3: Loop guard

Step 4: Initialization

Step 5: Moving through the matrices

Step 6: State before update statements

Step 7: State after update statements

Step 8: The update statements

The algorithm

The loop invariant, together with $\neg G$, must imply that we have computed the correct result, as stated by the postcondition:

Step	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** G **do**

⋮

	endwhile
2,3	$\left\{ \left(\left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg (G) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

ep	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
a	$\left\{ A = \widehat{A} \right\}$
4	
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**

⋮

	endwhile
,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg (m(A_{TL}) < m$
b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step	Annotated Algorithm: $A := \mathrm{LU}(A)$
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3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	
6	
8	
5b	
7	
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
	endwhile
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg(m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L\backslash U \wedge LU = \widehat{A} \right\}$

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Step 8: The update statements

The algorithm

The initialization, when started in a state where the precondition is *true* must put us in a state where the loop invariant is *true*:

ep	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
a	$\left\{ A = \widehat{A} \right\}$
4	
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**

⋮

	endwhile
3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg \left(m(A_{TL}) < m \right.$
b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

The initialization, when started in a state where the precondition is *true* must put us in a state where the loop invariant is *true*:

<table>
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Step	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	**Partition** $A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $U \rightarrow \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$ **where** A_{TL}, L_{TL}, and U_{TL} are 0×0
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	
6	
8	
5b	
7	
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
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1b	$\left\{ A = L\backslash U \wedge LU = \widehat{A} \right\}$

Operation: LU Factorization (LU)

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3 **while** $m(A_{TL}) < m(A)$ **do**

2,3 $\left\{ \left(\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$

5a

⋮

5b

⋮

endwhile

⋮

3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	**Repartition** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right), \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$ **where** α_{11}, 1, and υ_{11} are 1×1

⋮

5b	**Continue with** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right), \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$

⋮

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$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$, $\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right)$, $\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$
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The algorithm

At the top of the loop, the invariant is *true* and the matrices and/or vectors are repartitioned to expose submatrices and/or subvectors:

$\vdots$

while $m(A_{TL}) < m(A)$ **do**

$$\left\{ \left(\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$$

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$$

where α_{11}, 1, and υ_{11} are 1×1

$\vdots$

Now, substituting the exposed submatrices into the invariant

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array}$$

yields

$$\left(\begin{array}{c|c} A_{00} & \left(\begin{array}{c|c} a_{01} & A_{02} \end{array}\right) \\ \hline \left(\begin{array}{c} a_{10}^T \\ \hline A_{20} \end{array}\right) & \left(\begin{array}{c|c} \alpha_{11} & a_{21}^T \\ \hline a_{21} & A_{22} \end{array}\right) \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{00} & \left(\begin{array}{c|c} u_{01} & U_{02} \end{array}\right) \\ \hline \left(\begin{array}{c} l_{10}^T \\ \hline L_{20} \end{array}\right) & \left(\begin{array}{c|c} \widehat{\alpha}_{11} & \widehat{a}_{21}^T \\ \hline \widehat{a}_{21} & \widehat{A}_{22} \end{array}\right) - \left(\begin{array}{c} l_{10}^T \\ \hline L_{20} \end{array}\right) \left(\begin{array}{c|c} u_{01} & U_{02} \end{array}\right) \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}\left(\begin{array}{c|c} u_{01} & U_{02} \end{array}\right) = \left(\begin{array}{c|c} \widehat{a}_{01} & \widehat{A}_{02} \end{array}\right) \\ \hline \left(\begin{array}{c} l_{10}^T \\ \hline L_{20} \end{array}\right) U_{00} = \left(\begin{array}{c} \widehat{a}_{10}^T \\ \hline \widehat{A}_{20} \end{array}\right) & \end{array}$$

or, after some linear algebra,

$$
\left(\begin{array}{c|c|c}
A_{00} & a_{01} & A_{02} \\
\hline
a_{10}^T & \alpha_{11} & a_{12}^T \\
\hline
A_{20} & a_{21} & A_{22}
\end{array}\right)
=
\left(\begin{array}{c|c|c}
L\backslash_U 00 & u_{01} & U_{02} \\
\hline
l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\
\hline
L_{20} & \widehat{a}_{21} - L_{20} u_{01} & \widehat{A}_{22} - L_{20} U_{20}
\end{array}\right)
$$

$$
\wedge
\begin{array}{c|c|c}
L_{00} U_{00} = \widehat{A}_{00} & L_{00} u_{01} = \widehat{a}_{01} & L_{00} U_{02} = \widehat{A}_{02} \\
\hline
l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\
\hline
L_{20} U_{00} = \widehat{A}_{20} & &
\end{array}
$$

At the top of the loop, the invariant is *true* and the matrices and/or vectors are repartitioned to expose submatrices and/or subvectors:

$$\vdots$$

while $m(A_{TL}) < m(A)$ **do**

3 $$\left\{ \left(\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$$

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$$

where α_{11}, 1, and υ_{11} are 1×1

$$\left\{ \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c|c|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \widehat{a}_{21} - L_{20}u_{01} & \widehat{A}_{22} - L_{20}U_{20} \end{array} \right) \wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ \hline L_{20}U_{00} = \widehat{A}_{20} & & \end{array} \right\}$$

$$\vdots$$

Step	Annotated Algorithm: $A := \mathrm{LU}(A)$
1a	$\left\{ A = \hat{A} \right\}$
4	**Partition** $A \rightarrow \left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$, $L \rightarrow \left(\begin{array}{cc} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right)$, $U \rightarrow \left(\begin{array}{cc} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right)$ **where** A_{TL}, L_{TL}, and U_{TL} are 0×0
2	$\left\{ \left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{cc} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \hat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{cc} L_{TL}U_{TL} = \hat{A}_{TL} & L_{TL}U_{TR} = \hat{A}_{TR} \\ \hline L_{BL}U_{TL} = \hat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{cc} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \hat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{cc} L_{TL}U_{TL} = \hat{A}_{TL} & L_{TL}U_{TR} = \hat{A}_{TR} \\ \hline L_{BL}U_{TL} = \hat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	**Repartition** $\left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{ccc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{cc} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{ccc} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{cc} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{ccc} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ 0 & 0 & U_{22} \end{array} \right)$ **where** α_{11}, 1, and υ_{11} are 1×1
6	$\left\{ \left(\begin{array}{ccc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{ccc} L \backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \hat{\alpha} - l_{10}^T u_{01} & \hat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \hat{a}_{21} - L_{20}u_{01} & \hat{A}_{22} - L_{20}U_{20} \end{array} \right) \wedge \begin{array}{ccc} L_{00}U_{00} = \hat{A}_{00} & L_{00}u_{01} = \hat{a}_{01} & L_{00}U_{02} = \hat{A}_{02} \\ \hline l_{10}^T U_{00} = \hat{a}_{10}^T & & \\ \hline L_{20}U_{00} = \hat{A}_{20} & & \end{array} \right\}$
8	
5b	**Continue with** $\left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{ccc} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{cc} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \leftarrow \left(\begin{array}{ccc} L_{00} & 0 & 0 \\ l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{cc} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left(\begin{array}{ccc} U_{00} & u_{01} & U_{02} \\ 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$
7	
2	$\left\{ \left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{cc} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \hat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{cc} L_{TL}U_{TL} = \hat{A}_{TL} & L_{TL}U_{TR} = \hat{A}_{TR} \\ \hline L_{BL}U_{TL} = \hat{A}_{BL} & \end{array} \right\}$
	endwhile
2,3	$\left\{ \left(\left(\begin{array}{cc} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{cc} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \hat{A}_{BR} - L_{BL}U_{TR} \end{array} \right) \wedge \begin{array}{cc} L_{TL}U_{TL} = \hat{A}_{TL} & L_{TL}U_{TR} = \hat{A}_{TR} \\ \hline L_{BL}U_{TL} = \hat{A}_{BL} & \end{array} \right) \wedge \neg (m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \hat{A} \right\}$

Operation: LU Factorization (LU)

Goal oriented programming

The “worksheet”

Step 1: Precondition and postcondition

Step 2: Deriving the invariants

Step 3: Loop guard

Step 4: Initialization

Step 5: Moving through the matrices

Step 6: State before update statements

Step 7: State after update statements

Step 8: The update statements

The algorithm

At the bottom of the loop, the invariant must again be *true*. But the matrices and/or vectors are now repartitioned differently:

$\vdots$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$$

$$\left\{ \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$$

endwhile

$\vdots$

Substituting the exposed submatrices into the invariant

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right)$$

$$\wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array}$$

yields

$$\left(\begin{array}{c|c} \left(\begin{array}{c|c} A_{00} & a_{10} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) & \left(\begin{array}{c} A_{02}^T \\ \hline a_{21}^T \end{array}\right) \\ \hline \left(\begin{array}{c|c} A_{20} & a_{21} \end{array}\right) & A_{22} \end{array}\right) = \left(\begin{array}{c|c} \left(\begin{array}{c|c} L\backslash U_{00} & u_{01} \\ \hline l_{10}^T & \upsilon_{11} \end{array}\right) & \left(\begin{array}{c} U_{02}^T \\ \hline u_{21}^T \end{array}\right) \\ \hline \left(\begin{array}{c|c} L_{20} & l_{21} \end{array}\right) & \widehat{A}_{22} - \left(\begin{array}{c|c} L_{20} & l_{21} \end{array}\right)\left(\begin{array}{c} U_{02}^T \\ \hline u_{21}^T \end{array}\right) \end{array}\right)$$

$$\wedge \begin{array}{c|c} \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & 1 \end{array}\right)\left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline 0 & \upsilon_{11} \end{array}\right) = \left(\begin{array}{c|c} \widehat{A}_{00} & \widehat{a}_{01} \\ \hline \widehat{a}_{10}^T & \widehat{\alpha}_{11} \end{array}\right) & \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & 1 \end{array}\right)\left(\begin{array}{c} U_{02}^T \\ \hline u_{21}^T \end{array}\right) = \left(\begin{array}{c} \widehat{A}_{02}^T \\ \hline \widehat{a}_{21}^T \end{array}\right) \\ \hline \left(\begin{array}{c|c} L_{20} & l_{21} \end{array}\right)\left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline 0 & \upsilon_{11} \end{array}\right) = \left(\begin{array}{c|c} \widehat{A}_{20} & \widehat{a}_{21} \end{array}\right) & \end{array}$$

or, after some linear algebra,

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L\backslash_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20}U_{20} - l_{21}u_{12}^T \end{array}\right)$$

$$\wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20}U_{00} = \widehat{A}_{20} & L_{20}u_{01} + l_{21}\upsilon_{11} = \widehat{a}_{21} & \end{array}$$

At the bottom of the loop, the invariant must again be *true*. But the matrices and/or vectors are now repartitioned differently:

⋮

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$$

$$\left\{ \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20}U_{20} - l_{21}u_{12}^T \end{array}\right) \wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20}U_{00} = \widehat{A}_{20} & L_{20}u_{01} + l_{21}\upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$$

$$\left\{ \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$$

ndwhile

⋮

Step	**Annotated Algorithm:** $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	**Partition** $A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $U \rightarrow \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$ **where** A_{TL}, L_{TL}, and U_{TL} are 0×0
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	**Repartition** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$ **where** α_{11}, 1, and υ_{11} are 1×1
6	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c\|c\|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \widehat{a}_{21} - L_{20}u_{01} & \widehat{A}_{22} - L_{20}U_{20} \end{array}\right) \wedge \begin{array}{c\|c\|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ \hline L_{20}U_{00} = \widehat{A}_{20} & & \end{array} \right\}$
8	
5b	**Continue with** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$
7	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c\|c\|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20}U_{20} - l_{21}u_{12}^T \end{array}\right) \wedge \begin{array}{c\|c\|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20}U_{00} = \widehat{A}_{20} & L_{20}u_{01} + l_{21}\upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
	endwhile
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c\|c} L\backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL}U_{TR} \end{array}\right) \wedge \begin{array}{c\|c} L_{TL}U_{TL} = \widehat{A}_{TL} & L_{TL}U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL}U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg\left(m(A_{TL}) < m(A)\right) \right\}$
1b	$\left\{ A = L\backslash U \wedge LU = \widehat{A} \right\}$

Operation: LU Factorization (LU)

Goal oriented programming

The "worksheet"

Step 1: Precondition and postcondition

Step 2: Deriving the invariants

Step 3: Loop guard

Step 4: Initialization

Step 5: Moving through the matrices

Step 6: State before update statements

Step 7: State after update statements

Step 8: The update statements

The algorithm

To now determine what update to matrices and/or vectors must be made, one compares the state determined by Step 6 to the state determined by Step 7:

$$\vdots$$

$$\left\{ \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22}\end{array}\right) = \left(\begin{array}{c|c|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \widehat{a}_{21} - L_{20}u_{01} & \widehat{A}_{22} - L_{20}U_{20}\end{array}\right) \wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ \hline L_{20}U_{00} = \widehat{A}_{20} & & \end{array} \right\}$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR}\end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22}\end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR}\end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22}\end{array}\right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR}\end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22}\end{array}\right)$$

$$\left\{ \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22}\end{array}\right) = \left(\begin{array}{c|c|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20}U_{20} - l_{21}u_{12}^T\end{array}\right) \wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20}U_{00} = \widehat{A}_{20} & L_{20}u_{01} + l_{21}\upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$$

$$\vdots$$

To now determine what update to matrices and/or vectors must be made, one compares the state determined by Step 6 to the state determined by Step 7:

$$\vdots$$

$$\left\{ \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|c|c} L\backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ L_{20} & \widehat{a}_{21} - L_{20} u_{01} & \widehat{A}_{22} - L_{20} U_{20} \end{array}\right) \wedge \begin{array}{c|c|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ L_{20}U_{00} = \widehat{A}_{20} & & \end{array} \right\}$$

$a_{21} := a_{21} / \alpha_{11}$

$A_{22} := A_{22} - a_{21} a_{12}^T$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} L_{00} & 0 & 0 \\ l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{cc|c} U_{00} & u_{01} & U_{02} \\ 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$$

$$\left\{ \left(\begin{array}{cc|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{cc|c} L\backslash U_{00} & u_{01} & U_{02} \\ l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20}U_{20} - l_{21}u_{12}^T \end{array}\right) \wedge \begin{array}{cc|c} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} & L_{00}U_{02} = \widehat{A}_{02} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20}U_{00} = \widehat{A}_{20} & L_{20}u_{01} + l_{21}\upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$$

$$\vdots$$

Step	Annotated Algorithm: $A := \mathrm{LU}(A)$
1a	$\left\{ A = \widehat{A} \right\}$
4	**Partition** $A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$, $L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right)$, $U \rightarrow \left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right)$ **where** A_{TL}, L_{TL}, and U_{TL} are 0×0
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
3	**while** $m(A_{TL}) < m(A)$ **do**
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge m(A_{TL}) < m(A) \right\}$
5a	**Repartition** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$ **where** α_{11}, 1, and υ_{11} are 1×1
6	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c\|c\|c} L \backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \widehat{\alpha} - l_{10}^T u_{01} & \widehat{a}_{12}^T - l_{10}^T U_{02} \\ \hline L_{20} & \widehat{a}_{21} - L_{20} u_{01} & \widehat{A}_{22} - L_{20} U_{20} \end{array} \right) \wedge \begin{array}{c\|c\|c} L_{00} U_{00} = \widehat{A}_{00} & L_{00} u_{01} = \widehat{a}_{01} & L_{00} U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & & \\ \hline L_{20} U_{00} = \widehat{A}_{20} & & \end{array} \right\}$
8	$a_{21} := a_{21} / \alpha_{11}$ $A_{22} := A_{22} - a_{21} a_{12}^T$
5b	**Continue with** $\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array} \right)$, $\left(\begin{array}{c\|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c\|c\|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right)$
7	$\left\{ \left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left(\begin{array}{c\|c\|c} L \backslash U_{00} & u_{01} & U_{02} \\ \hline l_{10}^T & \upsilon_{11} & u_{12}^T \\ \hline L_{20} & l_{21} & \widehat{A}_{22} - L_{20} U_{20} - l_{21} u_{12}^T \end{array} \right) \wedge \begin{array}{c\|c\|c} L_{00} U_{00} = \widehat{A}_{00} & L_{00} u_{01} = \widehat{a}_{01} & L_{00} U_{02} = \widehat{A}_{02} \\ \hline l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + \upsilon_{11} = \widehat{\alpha}_{11} & l_{10}^T U_{02} + u_{12}^T = \widehat{a}_{12}^T \\ \hline L_{20} U_{00} = \widehat{A}_{20} & L_{20} u_{01} + l_{21} \upsilon_{11} = \widehat{a}_{21} & \end{array} \right\}$
2	$\left\{ \left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right\}$
	endwhile
2,3	$\left\{ \left(\left(\begin{array}{c\|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left(\begin{array}{c\|c} L \backslash U_{TL} & U_{TR} \\ \hline L_{BL} & \widehat{A}_{BR} - L_{BL} U_{TR} \end{array} \right) \wedge \begin{array}{c\|c} L_{TL} U_{TL} = \widehat{A}_{TL} & L_{TL} U_{TR} = \widehat{A}_{TR} \\ \hline L_{BL} U_{TL} = \widehat{A}_{BL} & \end{array} \right) \wedge \neg (m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Operation: LU Factorization (LU)

Goal oriented programming

The "worksheet"

Step 1: Precondition and postcondition

Step 2: Deriving the invariants

Step 3: Loop guard

Step 4: Initialization

Step 5: Moving through the matrices

Step 6: State before update statements

Step 7: State after update statements

Step 8: The update statements

The algorithm

Deleting all assertions leave the algorithm:

Algorithm: $A := \text{LU\_UNB\_VAR5}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$
where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

where α_{11} is 1×1

$a_{21} := a_{21}/\alpha_{11}$
$A_{22} := A_{22} - a_{21}a_{12}^T$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Repeat for the other invariants