Notes on Vector and Matrix Norms

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1 Absolute Value

Recall that if $\alpha \in \mathbb{C}$, then $|\alpha|$ equals its absolute value. In other words, if $\alpha = \alpha_r + i\alpha_c$, then $|\alpha| = \sqrt{\alpha_r^2 + \alpha_c^2} = \sqrt{\overline{\alpha}\alpha}$.

This absolute value function has the following properties:

- $\alpha \neq 0 \Rightarrow |\alpha| > 0$ ($| \cdot |$ is positive definite),
- $|\alpha\beta| = |\alpha||\beta|$ ($| \cdot |$ is homogeneous), and
- $|\alpha + \beta| \leq |\alpha| + |\beta|$ ($| \cdot |$ obeys the triangle inequality).

2 Vector Norms

A (vector) norm extends the notion of an absolute value (length or size) to vectors:

Definition 1. *Let $\nu : \mathbb{C}^n \to \mathbb{R}$. Then ν is a (vector) norm if for all $x, y \in \mathbb{C}^n$*

- *$x \neq 0 \Rightarrow \nu(x) > 0$ (ν is positive definite),*
- *$\nu(\alpha x) = |\alpha|\nu(x)$ (ν is homogeneous), and*
- *$\nu(x + y) \leq \nu(x) + \nu(y)$ (ν obeys the triangle inequality).*

Exercise 2. *Prove that if $\nu : \mathbb{C}^n \to \mathbb{R}$ is a norm, then $\nu(0) = 0$ (where the first 0 denotes the zero vector in $\mathbb{C}^n$).*

Answer: Let $x \in \mathbb{C}^n$ and $\vec{0}$ the zero vector of size n and 0 the scalar zero. Then

$$\begin{array}{rcll} \nu(\vec{0}) & = & \nu(0 \cdot x) & 0 \cdot x = \vec{0} \\ & = & |0|\nu(x) & \nu(\cdot) \text{ is homogeneous} \\ & = & 0 & \text{algebra} \end{array}$$

End of Answer

Note: often we will use $\| \cdot \|$ to denote a vector norm.

2.1 Vector 2-norm (length)

Definition 3. *The vector 2-norm* $\|\cdot\|_2 : \mathbb{C}^n \to \mathbb{R}$ *is defined by*

$$\|x\|_2 = \sqrt{x^H x} = \sqrt{\overline{\chi}_0\chi_0 + \cdots + \overline{\chi}_{n-1}\chi_{n-1}} = \sqrt{|\chi_0|^2 + \cdots + |\chi_{n-1}|^2}.$$

To show that the vector 2-norm is a norm, we will need the following theorem:

Theorem 4. *(Cauchy-Schartz inequality) Let* $x, y \in \mathbb{C}^n$*. Then* $|x^H y| \le \|x\|_2 \|y\|_2$*.*

Proof: Assume that $x \neq 0$ and $y \neq 0$, since otherwise the inequality is trivially true. We can then choose $\widehat{x} = x/\|x\|_2$ and $\widehat{y} = y/\|y\|_2$. This leaves us to prove that $|\widehat{x}^H\widehat{y}| \le 1$, with $\|\widehat{x}\|_2 = \|\widehat{y}\|_2 = 1$.

Pick $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ s that $\alpha\widehat{x}^H\widehat{y}$ is real and nonnegative. Note that since it is real, $\alpha\widehat{x}^H\widehat{y} = \overline{\alpha\widehat{x}^H\widehat{y}} = \overline{\alpha}\widehat{y}^H\widehat{x}$.

Now,

$$\begin{array}{rcll}
0 & \le & \|\widehat{x} - \alpha\widehat{y}\|_2^2 & \\
 & = & (x - \alpha\widehat{y})^H(\widehat{x} - \alpha\widehat{y}) & (\|z\|_2^2 = z^H z) \\
 & = & \widehat{x}^H\widehat{x} - \overline{\alpha}\widehat{y}^H\widehat{x} - \alpha\widehat{x}^H\widehat{y} + \overline{\alpha}\alpha\widehat{y}^H\widehat{y} & \text{(multiplying out)} \\
 & = & 1 - 2\alpha\widehat{x}^H\widehat{y} + |\alpha|^2 & (\|\widehat{x}\|_2 = \|\widehat{y}\|_2 = 1 \text{ and } \alpha\widehat{x}^H\widehat{y} = \overline{\alpha\widehat{x}^H\widehat{y}} = \overline{\alpha}\widehat{y}^H\widehat{x}) \\
 & = & 2 - 2\alpha\widehat{x}^H\widehat{y} & (|\alpha| = 1).
\end{array}$$

Thus $1 \ge \alpha\widehat{x}^H\widehat{y}$ and, taking the absolute value of both sides,

$$1 \ge |\alpha\widehat{x}^H\widehat{y}| = |\alpha||\widehat{x}^H\widehat{y}| = |\widehat{x}^H\widehat{y}|,$$

which is the desired result. QED

Theorem 5. *The vector 2-norm is a norm.*

Proof: To prove this, we merely check whether the three conditions are met:

Let $x, y \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_2 > 0$ ($\|\cdot\|_2$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then

  $$\|x\|_2 = \sqrt{|\chi_0|^2 + \cdots + |\chi_{n-1}|^2} \ge \sqrt{|\chi_j|^2} = |\chi_j| > 0.$$

- $\|\alpha x\|_2 = |\alpha|\|x\|_2$ ($\|\cdot\|_2$ is homogeneous):

  $\|\alpha x\|_2 = \sqrt{|\alpha\chi_0|^2 + \cdots + |\alpha\chi_{n-1}|^2} = \sqrt{|\alpha|^2|\chi_0|^2 + \cdots + |\alpha|^2|\chi_{n-1}|^2} = \sqrt{|\alpha|^2(|\chi_0|^2 + \cdots + |\chi_{n-1}|^2)} = |\alpha|\sqrt{|\chi_0|^2 + \cdots + |\chi_{n-1}|^2} = |\alpha|\|x\|_2$.

- $\|x + y\|_2 \le \|x\|_2 + \|y\|_2$ ($\|\cdot\|_2$ obeys the triangle inequality).

  $$\begin{array}{rcl}
  \|x + y\|_2^2 & = & (x + y)^H(x + y) \\
   & = & x^H x + y^H x + x^H y + y^H y \\
   & \le & \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 \\
   & = & (\|x\|_2 + \|y\|_2)^2.
  \end{array}$$

  Taking the square root of both sides yields the desired result.

QED

2.2 Vector 1-norm

Definition 6. *The vector 1-norm* $\| \cdot \|_1 : \mathbb{C}^n \to \mathbb{R}$ *is defined by*

$$\|x\|_1 = |\chi_0| + |\chi_1| + \cdots + |\chi_{n-1}|.$$

Exercise 7. *The vector 1-norm is a norm.*

Answer: We show that the three conditions are met:
Let $x, y \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_1 > 0$ ($\| \cdot \|_1$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then
  $$\|x\|_1 = |\chi_0| + \cdots + |\chi_{n-1}| \geq |\chi_j| > 0.$$

- $\|\alpha x\|_1 = |\alpha| \|x\|_1$ ($\| \cdot \|_1$ is homogeneous):

  $\|\alpha x\|_1 = |\alpha \chi_0| + \cdots + |\alpha \chi_{n-1}| = |\alpha||\chi_0| + \cdots + |\alpha||\chi_{n-1}| = |\alpha|(|\chi_0| + \cdots + |\chi_{n-1}|) = |\alpha|(|\chi_0| + \cdots + |\chi_{n-1}|) = |\alpha|\|x\|_1$.

- $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ ($\| \cdot \|_1$ obeys the triangle inequality).

$$\begin{array}{rcl}
\|x + y\|_1 & = & |\chi_0 + \psi_0| + |\chi_1 + \psi_1| + \cdots + |\chi_{n-1} + \psi_{n-1}| \\
& \leq & |\chi_0| + |\psi_0| + |\chi_1| + |\psi_1| + \cdots + |\chi_{n-1}| + |\psi_{n-1}| \\
& = & |\chi_0| + |\chi_1| + \cdots + |\chi_{n-1}| + |\psi_0| + |\psi_1| + \cdots + |\psi_{n-1}| \\
& = & \|x\|_1 + \|y\|_1.
\end{array}$$

End of Answer

The vector 1-norm is sometimes referred to as the "taxi-cab norm". It is the distance that a taxi travels along the streets of a city that has square blocks.

2.3 Vector ∞-norm (infinity norm)

Definition 8. *The vector* ∞*-norm* $\| \cdot \|_\infty : \mathbb{C}^n \to \mathbb{R}$ *is defined by* $\|x\|_\infty = \max_i |\chi_i|$.

Exercise 9. *The vector* ∞*-norm is a norm.*

Answer: We show that the three conditions are met:
Let $x, y \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $x \neq 0 \Rightarrow \|x\|_\infty > 0$ ($\| \cdot \|_\infty$ is positive definite):

  Notice that $x \neq 0$ means that at least one of its components is nonzero. Let's assume that $\chi_j \neq 0$. Then
  $$\|x\|_\infty = \max_i |\chi_i| \geq |\chi_j| > 0.$$

- $\|\alpha x\|_\infty = |\alpha|\|x\|_\infty$ ($\|\cdot\|_\infty$ is homogeneous):

  $$\|\alpha x\|_\infty = \max_i |\alpha\chi_i| = \max_i |\alpha||\chi_i| = |\alpha| \max_i |\chi_i| = |\alpha|\|x\|_\infty.$$

- $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$ ($\|\cdot\|_\infty$ obeys the triangle inequality).

$$\begin{aligned}
\|x+y\|_\infty &= \max_i |\chi_i + \psi_i| \\
&\le \max_i (|\chi_i| + |\psi_i|) \\
&\le \max_i (|\chi_i| + \max_j |\psi_j|) \\
&= \max_i |\chi_i| + \max_j |\psi_j| = \|x\|_\infty + \|y\|_\infty.
\end{aligned}$$

End of Answer

2.4 Vector p-norm

Definition 10. *The vector p-norm $\|\cdot\|_p : \mathbb{C}^n \to \mathbb{R}$ is defined by*

$$\|x\|_p = \sqrt[p]{|\chi_0|^p + |\chi_1|^p + \cdots + |\chi_{n-1}|^p}.$$

Proving that the p-norm is a norm is a little tricky and not particularly relevant to this course. To prove the triangle inequality requires the following classical result:

Theorem 11. *(Hölder inequality) Let $x, y \in \mathbb{C}^n$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \le p, q \le \infty$. Then $|x^H y| \le \|x\|_p \|y\|_q$.*

Clearly, the 1-norm and 2 norms are special cases of the p-norm. Also, $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$.

3 Matrix Norms

It is not hard to see that vector norms are all measures of how "big" the vectors are. Similarly, we want to have measures for how "big" matrices are. We will start with one that are somewhat artificial and then move on to the important class of induced matrix norms.

3.1 Frobenius norm

Definition 12. *The Frobenius norm $\|\cdot\|_F : \mathbb{C}^{m\times n} \to \mathbb{R}$ is defined by*

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\alpha_{i,j}|^2}.$$

Notice that one can think of the Frobenius norm as taking the columns of the matrix, stacking them on top of each other to create a vector of size $m \times n$, and then taking the vector 2-norm of the result.

Exercise 13. *Show that the Frobenius norm is a norm.*

Answer: The answer is to realize that if $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$ then

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\alpha_{i,j}|^2} = \sqrt{\sum_{j=0}^{n-1}\sum_{i=0}^{m-1}|\alpha_{i,j}|^2} = \sqrt{\sum_{j=0}^{n-1}\|a_j\|_2^2} = \sqrt{\left\| \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \right\|_2^2}.$$

In other words, it equals the vector 2-norm of the vector that is created by stacking the columns of A on top of each other. The fact that the Frobenius norm is a norm then comes from realizing this connection and exploiting it.

Alternatively, just grind through the three conditions! **End of Answer**

Similarly, other matrix norms can be created from vector norms by viewing the matrix as a vector. It turns out that other than the Frobenius norm, these aren't particularly interesting in practice.

3.2 Induced matrix norms

Definition 14. *Let $\|\cdot\|_\mu : \mathbb{C}^m \to \mathbb{R}$ and $\|\cdot\|_\nu : \mathbb{C}^n \to \mathbb{R}$ be vector norms. Define $\|\cdot\|_{\mu,\nu} : \mathbb{C}^{m\times n} \to \mathbb{R}$ by*

$$\|A\|_{\mu,\nu} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

Let us start by interpreting this. How "big" A is, as measured by $\|A\|_{\mu,\nu}$, is defined as the most that A magnifies the length of nonzero vectors, where the length of the vectors (x) is measured with norm $\|\cdot\|_\nu$ and the length of the transformed vector (Ax) is measured with norm $\|\cdot\|_\mu$.

Two comments are in order. First,

$$\sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} = \sup_{\|x\|_\nu = 1} \|Ax\|_\mu.$$

This follows immediately from the fact this sequence of equivalences:

$$\sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \|\frac{Ax}{\|x\|_\nu}\|_\mu = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \|A\frac{x}{\|x\|_\nu}\|_\mu = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0 \\ y = \frac{x}{\|x\|_\nu}}} \|Ay\|_\mu = \sup_{\|y\|_\nu = 1} \|Ay\|_\mu = \sup_{\|x\|_\nu = 1} \|Ax\|_\mu.$$

Also the "sup" (which stands for supremum) is used because we can't claim yet that there is a vector x with $\|x\|_\nu = 1$ for which

$$\|A\|_{\mu,\nu} = \|Ax\|_\mu.$$

The fact is that there is always such a vector x. The proof depends on a result from real analysis (sometimes called "advanced calculus") that states that $\sup_{x \in S} f(x)$ is attained for some vector $x \in S$ as long as f is continuous and S is a compact set. Since real analysis is not a prerequisite for this course, the reader may have to take this on faith!

We conclude that the following two definitions are equivalent definitions to the one we already gave:

Definition 15. *Let $\|\cdot\|_\mu : \mathbb{C}^m \to \mathbb{R}$ and $\|\cdot\|_\nu : \mathbb{C}^n \to \mathbb{R}$ be vector norms. Define $\|\cdot\|_{\mu,\nu} : \mathbb{C}^{m\times n} \to \mathbb{R}$ by*

$$\|A\|_{\mu,\nu} = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

and

Definition 16. *Let $\|\cdot\|_\mu : \mathbb{C}^m \to \mathbb{R}$ and $\|\cdot\|_\nu : \mathbb{C}^n \to \mathbb{R}$ be vector norms. Define $\|\cdot\|_{\mu,\nu} : \mathbb{C}^{m\times n} \to \mathbb{R}$ by*

$$\|A\|_{\mu,\nu} = \max_{\|x\|_\nu = 1} \|Ax\|_\mu.$$

Theorem 17. *$\|\cdot\|_{\mu,\nu} : \mathbb{C}^{m\times n} \to \mathbb{R}$ is a norm.*

Proof: To prove this, we merely check whether the three conditions are met:
Let $A, B \in \mathbb{C}^{m\times n}$ and $\alpha \in \mathbb{C}$ be arbitrarily chosen. Then

- $A \neq 0 \Rightarrow \|A\|_{\mu,\nu} > 0$ ($\|\cdot\|_{\mu,\nu}$ is positive definite):

  Notice that $A \neq 0$ means that at least one of its columns is not a zero vector (since at least one element). Let us assume it is the jth column, a_j, that is nonzero. Then

  $$\|A\|_{\mu,\nu} = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} \geq \frac{\|Ae_j\|_\mu}{\|e_j\|_\nu} = \frac{\|a_j\|_\mu}{\|e_j\|_\nu} > 0.$$

- $\|\alpha A\|_{\mu,\nu} = |\alpha|\|A\|_{\mu,\nu}$ ($\|\cdot\|_{\mu,\nu}$ is homogeneous):

  $$\|\alpha A\|_{\mu,\nu} = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|\alpha Ax\|_\mu}{\|x\|_\nu} = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} |\alpha| \frac{\|Ax\|_\mu}{\|x\|_\nu} = |\alpha| \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} = |\alpha|\|A\|_{\mu,\nu}.$$

- $\|A+B\|_{\mu,\nu} \leq \|A\|_{\mu,\nu} + \|B\|_{\mu,\nu}$ ($\|\cdot\|_{\mu,\nu}$ obeys the triangle inequality).

$$\begin{aligned}\|A+B\|_{\mu,\nu} &= \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|(A+B)x\|_\mu}{\|x\|_\nu} = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax+Bx\|_\mu}{\|x\|_\nu} \leq \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu + \|Bx\|_\mu}{\|x\|_\nu} \\ &\leq \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \left(\frac{\|Ax\|_\mu}{\|x\|_\nu} + \frac{\|Bx\|_\mu}{\|x\|_\nu} \right) \leq \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_\mu}{\|x\|_\nu} + \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Bx\|_\mu}{\|x\|_\nu} = \|A\|_{\mu,\nu} + \|B\|_{\mu,\nu}.\end{aligned}$$

QED

3.3 Special cases used in practice

The most important case of $\|\cdot\|_{\mu,\nu} : \mathbb{C}^{m\times n} \to \mathbb{R}$ uses the same norm for $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ (except that m may not equal n).

Definition 18. *Define* $\|\cdot\|_p : \mathbb{C}^{m\times n} \to \mathbb{R}$ *by*

$$\|A\|_p = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p.$$

Theorem 19. *Let* $A \in \mathbb{C}^{m\times n}$ *and partition* $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$. *Show that*

$$\|A\|_1 = \max_{0\leq j<n} \|a_j\|_1.$$

Proof: Let $\bar{\jmath}$ be chosen so that $\max_{0\leq j<n} \|a_j\|_1 = \|a_{\bar{\jmath}}\|_1$. Then

$$\begin{array}{rcl}
\max_{\|x\|_1=1} \|Ax\|_1 & = & \max_{\|x\|_1=1} \left\| \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{array} \right) \right\|_1 \\
& = & \max_{\|x\|_1=1} \|\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}\|_1 \\
& \leq & \max_{\|x\|_1=1} \left(\|\chi_0 a_0\|_1 + \|\chi_1 a_1\|_1 + \cdots + \|\chi_{n-1} a_{n-1}\|_1 \right) \\
& = & \max_{\|x\|_1=1} \left(|\chi_0| \|a_0\|_1 + |\chi_1| \|a_1\|_1 + \cdots + |\chi_{n-1}| \|a_{n-1}\|_1 \right) \\
& \leq & \max_{\|x\|_1=1} \left(|\chi_0| \|a_{\bar{\jmath}}\|_1 + |\chi_1| \|a_{\bar{\jmath}}\|_1 + \cdots + |\chi_{n-1}| \|a_{\bar{\jmath}}\|_1 \right) \\
& = & \max_{\|x\|_1=1} \left(|\chi_0| + |\chi_1| + \cdots + |\chi_{n-1}| \right) \|a_{\bar{\jmath}}\|_1 \\
& = & \|a_{\bar{\jmath}}\|_1.
\end{array}$$

Also,

$$\|a_{\bar{\jmath}}\|_1 = \|Ae_{\bar{\jmath}}\|_1 \leq \max_{\|x\|_1=1} \|Ax\|_1.$$

Hence

$$\|a_{\bar{\jmath}}\|_1 \leq \max_{\|x\|_1=1} \|Ax\|_1 \leq \|a_{\bar{\jmath}}\|_1$$

which implies that

$$\max_{\|x\|_1=1} \|Ax\|_1 = \|a_{\bar{\jmath}}\|_1 = \max_{0\leq j<n} \|a_j\|.$$

QED

Exercise 20. *Let* $A \in \mathbb{C}^{m\times n}$ *and partition* $A = \left(\begin{array}{c} \widehat{a}_0^T \\ \hline \widehat{a}_1^T \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T \end{array} \right)$. *Show that*

$$\|A\|_\infty = \max_{0\leq i<m} \|\widehat{a}_i\|_1 = \max_{0\leq i<m} \left(|\alpha_{i,0}| + |\alpha_{i,1}| + \cdots + |\alpha_{i,n-1}| \right)$$

Answer: Partition $A = \left(\begin{array}{c} \widehat{a}_0^T \\ \hline \widehat{a}_1^T \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T \end{array} \right)$. Then

$$
\begin{aligned}
\|A\|_\infty &= \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{\|x\|_\infty=1} \left\| \left(\begin{array}{c} \widehat{a}_0^T \\ \hline \widehat{a}_1^T \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T \end{array} \right) x \right\|_\infty = \max_{\|x\|_\infty=1} \left\| \left(\begin{array}{c} \widehat{a}_0^T x \\ \hline \widehat{a}_1^T x \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T x \end{array} \right) \right\|_\infty \\
&= \max_{\|x\|_\infty=1} \left(\max_i |a_i^T x| \right) = \max_{\|x\|_\infty=1} \max_i | \sum_{p=0}^{n-1} \alpha_{i,p}\chi_p| \le \max_{\|x\|_\infty=1} \max_i \sum_{p=0}^{n-1} |\alpha_{i,p}\chi_p| \\
&= \max_{\|x\|_\infty=1} \max_i \sum_{p=0}^{n-1} (|\alpha_{i,p}||\chi_p|) \le \max_{\|x\|_\infty=1} \max_i \sum_{p=0}^{n-1} (|\alpha_{i,p}|(\max_k |\chi_k|)) \le \max_{\|x\|_\infty=1} \max_i \sum_{p=0}^{n-1} (|\alpha_{i,p}|\|x\|_\infty) \\
&= \max_i \sum_{p=0}^{n-1} (|\alpha_{i,p}| = \|\widehat{a}_i\|_1
\end{aligned}
$$

so that $\|A\|_\infty \le \max_i \|\widehat{a}_i\|_1$.

We also want to show that $\|A\|_\infty \ge \max_i \|\widehat{a}_i\|_1$. Let k be such that $\max_i \|\widehat{a}_i\|_1 = \|\widehat{a}_k\|_1$ and pick $y = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$ so that $\widehat{a}_k^T y = |\alpha_{k,0}| + |\alpha_{k,1}| + \cdots + |\alpha_{k,n-1}| = \|\widehat{a}_k\|_1$. (This is a matter of picking ψ_i so that $|\psi_i| = 1$ and $\psi_i \alpha_{k,i} = |\alpha_{k,i}|$.) Then

$$
\begin{aligned}
\|A\|_\infty = \max_{\|x\|_1=1} \|Ax\|_\infty &= \max_{\|x\|_1=1} \left\| \left(\begin{array}{c} \widehat{a}_0^T \\ \hline \widehat{a}_1^T \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T \end{array} \right) x \right\|_\infty \ge \left\| \left(\begin{array}{c} \widehat{a}_0^T \\ \hline \widehat{a}_1^T \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T \end{array} \right) y \right\|_\infty \\
&= \left\| \left(\begin{array}{c} \widehat{a}_0^T y \\ \hline \widehat{a}_1^T y \\ \hline \vdots \\ \hline \widehat{a}_{m-1}^T y \end{array} \right) \right\|_\infty \ge |\widehat{a}_k^T y| = \widehat{a}_k^T y = \|\widehat{a}_k\|_1. = \max_i \|\widehat{a}_i\|_1
\end{aligned}
$$

End of Answer

Notice that in the above exercise $\widehat{a}_i$ is really $(\widehat{a}_i^T)^T$ since $\widehat{a}_i^T$ is the label for the ith row of matrix A.

3.4 Discussion

While $\|\cdot\|_2$ is a very important matrix norm, it is in practice difficult to compute. The matrix norms, $\|\cdot\|_F$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$ are more easily computed and hence more practical in many instances.

3.5 Submultiplicative norms

Definition 21. *A matrix norm $\| \cdot \|_\nu : \mathbb{C}^{m \times n} \to \mathbb{R}$ is said to be submultiplicative (consistent) if it also satisfies*

$$\|AB\|_\nu \leq \|A\|_\nu \|B\|_\nu.$$

Theorem 22. *Let $\| \cdot \|_\nu : \mathbb{C}^n \to \mathbb{R}$ be a vector norm and given any matrix $C \in \mathbb{C}^{m \times n}$ define the corresponding induced matrix norm as*

$$\|C\|_\nu = \max_{x \neq 0} \frac{\|Cx\|_\nu}{\|x\|_\nu} = \max_{\|x\|_\nu = 1} \|Cx\|_\nu.$$

Then for any $A \in \mathbb{C}^{m \times k}$ and $B \in \mathbb{C}^{k \times n}$ the inequality $\|AB\|_\nu \leq \|A\|_\nu \|B\|_\nu$ holds.

In other words, induced matrix norms are submultiplicative.

To prove the above, it helps to first proof a simpler result:

Lemma 23. *Let $\| \cdot \|_\nu : \mathbb{C}^n \to \mathbb{R}$ be a vector norm and given any matrix $C \in \mathbb{C}^{m \times n}$ define the induced matrix norm as*

$$\|C\|_\nu = \max_{x \neq 0} \frac{\|Cx\|_\nu}{\|x\|_\nu} = \max_{\|x\|_\nu = 1} \|Cx\|_\nu.$$

Then for any $A \in \mathbb{C}^{m \times n}$ and $y \in \mathbb{C}^n$ the inequality $\|Ay\|_\nu \leq \|A\|_\nu \|y\|_\nu$ holds.

Proof: If $y = 0$, the result obviously holds since then $\|Ay\|_\nu = 0$ and $\|y\|_\nu = 0$. Let $y \neq 0$. Then

$$\|A\|_\nu = \max_{x \neq 0} \frac{\|Ax\|_\nu}{\|x\|_\nu} \geq \frac{\|Ay\|_\nu}{\|y\|_\nu}.$$

Rearranging this yields $\|Ay\|_\nu \leq \|A\|_\nu \|y\|_\nu$. QED

We can now prove the theorem:

Proof:

$$\|AB\|_\nu = \max_{\|x\|_\nu = 1} \|ABx\|_\nu = \max_{\|x\|_\nu = 1} \|A(Bx)\|_\nu \leq \max_{\|x\|_\nu = 1} \|A\|_\nu \|Bx\|_\nu \leq \max_{\|x\|_\nu = 1} \|A\|_\nu \|B\|_\nu \|x\|_\nu = \|A\|_\nu \|B\|_\nu.$$

QED

Exercise 24. *Show that $\|Ax\|_\mu \leq \|A\|_{\mu,\nu} \|x\|_\nu$.*

Answer: W.l.o.g. let $x \neq 0$.

$$\|A\|_{\mu,\nu} = \max_{y \neq 0} \frac{\|Ay\|_\mu}{\|y\|_\nu} \geq \frac{\|Ax\|_\mu}{\|x\|_\nu}.$$

Rearranging this establishes the result. **End of Answer**

Exercise 25. *Show that $\|AB\|_\mu \leq \|A\|_{\mu,\nu} \|B\|_\nu$.*

Answer:

$$\|AB\|_{\mu,\nu} = \max_{\|x\|_\nu = 1} \|ABx\|_\mu \leq \max_{\|x\|_\nu = 1} \|A\|_{\mu,\nu} \|Bx\|_\nu = \|A\|_{\mu,\nu} \max_{\|x\|_\nu = 1} \|Bx\|_\nu = \|A\|_{\mu,\nu} \|B\|_\nu$$

End of Answer

Exercise 26. *Show that the Frobenius norm, $\| \cdot \|_F$, is submultiplicative.*

Answer:

$$
\begin{array}{rcl}
\|AB\|_F^2 & = & \left\| \left(\begin{array}{c} \widehat a_0^H \\ \widehat a_1^H \\ \vdots \\ \widehat a_{m-1}^H \end{array} \right) \left(\begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) \right\|_F^2 = \left\| \left(\begin{array}{c|c|c|c} \widehat a_0^H b_0 & \widehat a_0^H b_1 & \cdots & \widehat a_0^H b_{n-1} \\ \hline \widehat a_0^H b_0 & \widehat a_0^H b_1 & \cdots & \widehat a_0^H b_{n-1} \\ \hline \vdots & \vdots & & \vdots \\ \hline \widehat a_{m-1}^H b_0 & \widehat a_{m-1}^H b_1 & \cdots & \widehat a_{m-1}^H b_{n-1} \end{array} \right) \right\|_F^2 = \sum_i \sum_j |\widehat a_i^H b_j|^2 \\
& \leq & \sum_i \sum_j \|\widehat a_i^H\|_2^2 \|b_j\|^2 \qquad \text{(Cauchy-Schwartz)} \\
& = & \left(\sum_i \|\widehat a_i\|_2^2 \right) \left(\sum_j \|b_j\|^2 \right) = \left(\sum_i \widehat a_i^H \widehat a_i \right) \left(\sum_j b_j^H b_j \right) \\
& \leq & \left(\sum_i \sum_j |\widehat a_i^H \widehat a_j| \right) \left(\sum_i \sum_j |b_i^H b_j| \right) = \|A\|_F^2 \|B\|_F^2 .
\end{array}
$$

so that $\|AB\|_F^2 \leq \|A\|_2^2 \|B\|_2^2$. Taking the square-root of both sides established the desired result.
End of Answer

4 An Application to Conditioning of Linear Systems

A question we will run into later in the course asks how accurate we can expect the solution of a linear system to be if the right-hand side of the system has error in it.

Formally, this can be stated as follows: We wish to solve $Ax = b$, where $A \in \mathbb{C}^{m\times m}$ but the right-hand side has been perturbed by a small vector so that it becomes $b + \delta\!b$. (Notice how that δ touches the b. This is meant to convey that this is a symbol that represents a vector rather than the vector b that is multiplied by a scalar δ.) The question now is how a relative error in b propogates into a potential error in the solution x.

This is summarized as follows:

$$
\begin{array}{rcll}
Ax & = & b & \qquad \text{Exact equation} \\
A(x + \delta\!x) & = & b + \delta\!b & \qquad \text{Perturbed equation}
\end{array}
$$

We would like to determine a formula, $\kappa(A, b, \delta\!b)$, that tells us how much a relative error in b is potentially amplified into an error in the solution b:

$$
\frac{\|\delta\!x\|}{\|x\|} \leq \kappa(A, b, \delta\!b) \frac{\|\delta\!b\|}{\|b\|}.
$$

We will assume that A has an inverse. To find an expression for $\kappa(A, b, \delta\!b)$, we notice that

$$
\begin{array}{rcl}
Ax + A\delta\!x & = & b + \delta\!b \\
Ax & = & b \qquad - \\
\hline
A\delta\!x & = & \delta\!b
\end{array}
$$

and from this

$$
\begin{array}{rcl}
Ax & = & b \\
\delta\!x & = & A^{-1}\delta\!b.
\end{array}
$$

If we now use a vector norm $\|\cdot\|$ and induced matrix norm $\|\cdot\|$, then

$$
\begin{array}{rcl}
\|b\| & = & \|Ax\| \leq \|A\|\|x\| \\
\|\delta\!x\| & = & \|A^{-1}\delta\!b\| \leq \|A^{-1}\|\|\delta\!b\|.
\end{array}
$$

From this we conclude that

$$\begin{aligned} \frac{1}{\|x\|} &\leq \|A\| \frac{1}{\|b\|} \\ \|\delta x\| &\leq \|A^{-1}\| \|\delta b\|. \end{aligned}$$

so that

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}.$$

Thus, the desired expression $\kappa(A, b, \delta b)$ doesn't depend on anything but the matrix A:

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \|A^{-1}\|}_{\kappa(A)} \frac{\|\delta b\|}{\|b\|}.$$

$\kappa(A) = \|A\| \|A^{-1}\|$ the called the *condition number* of matrix A.

A question becomes whether this is a pessimistic result or whether there are examples of b and δb for which the relative error in b is amplified by exactly $\kappa(A)$. The answer is, unfortunately, "yes!", as we will show next.

Notice that

- There is an $\widehat{x}$ for which

  $$\|A\| = \max_{\|x\|=1} \|Ax\| = \|A\widehat{x}\|,$$

  namely the x for which the maximum is attained. Pick $\widehat{b} = A\widehat{x}$.

- There is an $\widehat{\delta b}$ for which

  $$\|A^{-1}\| = \max_{\|x\| \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \frac{\|A^{-1}\widehat{\delta b}\|}{\|\widehat{\delta b}\|},$$

  again, the x for which the maximum is attained.

It is when solving the perturbed system

$$A(x + \delta x) = \widehat{b} + \widehat{\delta b}$$

that the maximal magnification by $\kappa(A)$ is attained.

Exercise 27. *Let $\| \cdot \|$ be a matrix norm induced by the $\| \cdots \|$ vector norm. Show that $\kappa(A) = \|A\| \|A^{-1}\| \geq 1$.*

Answer:

$$\|I\| = \|AA^{-1}\| \leq \|A\| \|A^{-1}\|.$$

But

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|} \|x\| = 1.$$

Hence $1 \leq \|A\| \|A^{-1}\|$. **End of Answer**

This last exercise shows that there will always be choices for b and δb for which the relative error is at best directly translated into an equal relative error in the solution (if $\kappa(A) = 1$).

5 Equivalence of Norms

Many results we encounter show that the norm of a particular vector or matrix is small. Obviously, it would be unfortunate if a vector or matrix is large in one norm and small in another norm. The following result shows that, modulo a constant, all norms are equivalent. Thus, if the vector is small in one norm, it is small in other norms as well.

Theorem 28. *Let* $\|\cdot\|_\mu : \mathbb{C}^n \to \mathbb{R}$ *and* $\|\cdot\|_\nu : \mathbb{C}^n \to \mathbb{R}$ *be vector norms. Then there exist constants* $\alpha_{\mu,\nu}$ *and* $\beta_{\mu,\nu}$ *such that for all* $x \in \mathbb{C}^n$

$$\alpha_{\mu,\nu}\|x\|_\mu \leq \|x\|_\nu \leq \beta_{\mu,\nu}\|x\|_\mu.$$

A similar result holds for matrix norms:

Theorem 29. *Let* $\|\cdot\|_\mu : \mathbb{C}^{m\times n} \to \mathbb{R}$ *and* $\|\cdot\|_\nu : \mathbb{C}^{m\times n} \to \mathbb{R}$ *be matrix norms. Then there exist constants* $\alpha_{\mu,\nu}$ *and* $\beta_{\mu,\nu}$ *such that for all* $A \in \mathbb{C}^{m\times n}$

$$\alpha_{\mu,\nu}\|A\|_\mu \leq \|A\|_\nu \leq \beta_{\mu,\nu}\|A\|_\mu.$$