Notes on the Singular Value Decomposition

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September 11, 2014

NOTE: I have not thoroughly proof-read these notes!!!

We recommend the reader review Weeks 9-11 of Linear Algebra: Foundations to Frontiers - Notes to LAFF With.

1 Orthogonality and Unitary Matrices

Definition 1. *Let $u, v \in \mathbb{C}^m$. These vectors are orthogonal (perpendicular) if $u^H v = 0$.*

Definition 2. *Let $q_0, q_1, \ldots, q_{n-1} \in \mathbb{C}^m$. These vectors are said to be mutually orthonormal if for all $0 \leq i, j < n$*

$$q_i^H q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Notice that for n vectors of length m to be mutually orthonormal, n must be less than or equal to m. This is because n mutually orthonormal vectors are linearly independent and there can be at most m linearly independent vectors of length m.

Definition 3. *Let $Q \in \mathbb{C}^{m \times n}$ (with $n \leq m$). Then Q is said to be an orthonormal matrix if $Q^H Q = I$ (the identity).*

Exercise 4. *Let $Q \in \mathbb{C}^{m \times n}$ (with $n \leq m$). Partition $Q = \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{n-1} \end{array} \right)$. Show that Q is an orthonormal matrix if and only if $q_0, q_1, \ldots, q_{n-1}$ are mutually orthonormal.*

Definition 5. *Let $Q \in \mathbb{C}^{m \times m}$. Then Q is said to be a unitary matrix if $Q^H Q = I$ (the identity).*

Notice that unitary matrices are always square and only square matrices can be unitary. Sometimes the term *orthogonal matrix* is used instead of unitary matrix, especially if the matrix is real valued.

Exercise 6. *Let $Q \in \mathbb{C}^{m \times m}$. Show that if Q is unitary then $Q^{-1} = Q^H$ and $QQ^H = I$.*

Exercise 7. *Let $Q_0, Q_1 \in \mathbb{C}^{m \times m}$ both be unitary. Show that their product, $Q_0 Q_1$, is unitary.*

Exercise 8. *Let $Q_0, Q_1, \ldots, Q_{k-1} \in \mathbb{C}^{m \times m}$ all be unitary. Show that their product, $Q_0 Q_1 \cdots Q_{k-1}$, is unitary.*

The following is a very important observation: Let Q be a unitary matrix with

$$Q = \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{m-1} \end{array} \right).$$

Let $x \in \mathbb{C}^m$. Then

$$\begin{aligned}
x &= QQ^H x = \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{m-1} \end{array} \right) \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{m-1} \end{array} \right)^H x \\
&= \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{m-1} \end{array} \right) \left(\begin{array}{c} q_0^H \\ q_1^H \\ \vdots \\ q_{m-1}^H \end{array} \right) x \\
&= \left(\begin{array}{c|c|c|c} q_0 & q_1 & \cdots & q_{m-1} \end{array} \right) \left(\begin{array}{c} q_0^H x \\ q_1^H x \\ \vdots \\ q_{m-1}^H x \end{array} \right) \\
&= (q_0^H x) q_0 + (q_1^H x) q_1 + \cdots + (q_{m-1}^H x) q_{m-1}.
\end{aligned}$$

What does this mean?

- The vector $x = \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{m-1} \end{array} \right)$ gives the coefficients when the vector x is written as a linear combination of the unit basis vectors:
$$x = \chi_0 e_0 + \chi_1 e_1 + \cdots + \chi_{m-1} e_{m-1}.$$
- The vector
$$Q^H x = \left(\begin{array}{c} q_0^H x \\ q_1^H x \\ \vdots \\ q_{m-1}^H x \end{array} \right)$$
gives the coefficients when the vector x is written as a linear combination of the orthonormal vectors $q_0, q_1, \ldots, q_{m-1}$:
$$x = (q_0^H x) q_0 + (q_1^H x) q_1 + \cdots + (q_{m-1}^H x) q_{m-1}.$$
- The vector $(q_i^H x) q_i$ equals the component of x in the direction of vector q_i.

Another way of looking at this is that if $q_0, q_1, \ldots, q_{m-1}$ is an orthonormal basis for $\mathbb{C}^m$, then any $x \in \mathbb{C}^m$ can be written as a linear combination of these vectors:

$$x = \alpha_0 q_0 + \alpha_1 q_1 + \cdots + \alpha_{m-1} q_{m-1}.$$

Now,

$$\begin{aligned}
q_i^H x &= q_i^H (\alpha_0 q_0 + \alpha_1 q_1 + \cdots + \alpha_{i-1} q_{i-1} + \alpha_i q_i + \alpha_{i+1} q_{i+1} + \cdots + \alpha_{m-1} q_{m-1}) \\
&= \alpha_0 \underbrace{q_i^H q_0}_{0} + \alpha_1 \underbrace{q_i^H q_1}_{0} + \cdots + \alpha_{i-1} \underbrace{q_i^H q_{i-1}}_{0} + \alpha_i \underbrace{q_i^H q_i}_{1} + \alpha_{i+1} \underbrace{q_i^H q_{i+1}}_{0} + \cdots + \alpha_{m-1} \underbrace{q_i^H q_{m-1}}_{0} \\
&= \alpha_i.
\end{aligned}$$

Thus $q_i^H x = \alpha_i$, the coefficient that multiplies q_i.

Exercise 9. *Let* $U \in \mathbb{C}^{m\times m}$ *be unitary and* $x \in \mathbb{C}^m$*, then* $\|Ux\|_2 = \|x\|_2$.

Exercise 10. *Let* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{n\times n}$ *be unitary matrices and* $A \in \mathbb{C}^{m\times n}$*. Then* $\|UA\|_2 = \|AV\|_2 = \|A\|_2$.

Exercise 11. *Let* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{n\times n}$ *be unitary matrices and* $A \in \mathbb{C}^{m\times n}$*. Then* $\|UA\|_F = \|AV\|_F = \|A\|_F$.

2 Toward the SVD

Lemma 12. *Given* $A \in \mathbb{C}^{m\times n}$ *there exists unitary* $U \in \mathbb{C}^{m\times m}$*, unitary* $V \in \mathbb{C}^{n\times n}$*, and diagonal* $D \in \mathbb{R}^{m\times n}$ *such that* $A = UDV^H$ *where* $D = \left(\begin{array}{c|c} D_{TL} & 0 \\ \hline 0 & 0 \end{array}\right)$ *with* $D_{TL} = \mathrm{diag}(\delta_0, \cdots, \delta_{r-1})$ *and* $\delta_i > 0$ *for* $0 \le i < r$.

Proof: First, let us observe that if $A = 0$ (the zero matrix) then the theorem trivially holds: $A = UDV^H$ where $U = I_{m\times m}$, $V = I_{n\times n}$, and $D = \left(\begin{array}{c|c} & \\ \hline & 0 \end{array}\right)$, so that D_{TL} is 0×0. Thus, w.l.o.g. assume that $A \neq 0$.

We will prove this for $m \ge n$, leaving the case where $m \le n$ as an exercise, employing a proof by induction on n.

- **Base case:** $n = 1$. In this case $A = \left(a_0 \right)$ where $a_0 \in \mathbb{R}^m$ is its only column. By assumption, $a_0 \neq 0$. Then

$$A = \left(a_0 \right) = \left(u_0 \right) (\|a_0\|_2) \left(1 \right)^H$$

where $u_0 = a_0/\|a_0\|_2$. Choose $U_1 \in \mathbb{C}^{m\times(m-1)}$ so that $U = \left(u_0 \mid U_1 \right)$ is unitary. Then

$$A = \left(a_0 \right) = \left(u_0 \right) (\|a_0\|_2) \left(1 \right)^H = \left(u_0 \mid U_1 \right) \left(\begin{array}{c|c} \|a_0\|_2 & \\ \hline 0 & \end{array}\right) \left(1 \right)^H = UDV^H$$

where $D_{TL} = \left(\delta_0 \right) = \left(\|a_0\|_2 \right)$ and $V = \left(1 \right)$.

- **Inductive step:** Assume the result is true for all matrices with $1 \le k < n$ columns. Show that it is true for matrices with n columns.

  Let $A \in \mathbb{C}^{m\times n}$ with $n \ge 2$. W.l.o.g., $A \neq 0$ so that $\|A\|_2 \neq 0$. Let δ_0 and $v_0 \in \mathbb{C}^n$ have the property that $\|v_0\|_2 = 1$ and $\delta_0 = \|Av_0\|_2 = \|A\|_2$. (In other words, v_0 is the vector that maximizes $\max_{\|x\|_2=1} \|Ax\|_2$.) Let $u_0 = Av_0/\delta_0$. Note that $\|u_0\|_2 = 1$. Choose $U_1 \in \mathbb{C}^{m\times(m-1)}$ and $V_1 \in \mathbb{C}^{n\times(n-1)}$ so that $\tilde{U} = \left(u_0 \mid U_1 \right)$ and $\tilde{V} = \left(v_0 \mid V_1 \right)$ are unitary. Then

$$\begin{aligned} \tilde{U}^H A \tilde{V} &= \left(u_0 \mid U_1 \right)^H A \left(v_0 \mid V_1 \right) \\ &= \left(\begin{array}{c|c} u_0^H A v_0 & u_0^H A V_1 \\ \hline U_1^H A v_0 & U_1^H A V_1 \end{array}\right) = \left(\begin{array}{c|c} \delta_0 u_0^H u_0 & u_0^H A V_1 \\ \hline \delta U_1^H u_0 & U_1^H A V_1 \end{array}\right) = \left(\begin{array}{c|c} \delta_0 & w^H \\ \hline 0 & B \end{array}\right), \end{aligned}$$

  where $w = V_1^H A^H u_0$ and $B = U_1^H A V_1$. Now, we will argue that $w = 0$, the zero vector of appropriate size:

$$\delta_0^2 = \|A\|_2^2 \quad = \quad \|U^H A V\|_2^2 = \max_{x\neq 0} \frac{\|U^H A V x\|_2^2}{\|x\|_2^2} = \max_{x\neq 0} \frac{\left\| \left(\begin{array}{c|c} \delta_0 & w^H \\ \hline 0 & B \end{array}\right) x \right\|_2^2}{\|x\|_2^2}$$

$$\geq \quad \frac{\left\| \left(\begin{array}{c|c} \delta_0 & w^H \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c} \delta_0 \\ \hline w \end{array} \right) \right\|_2^2}{\left\| \left(\begin{array}{c} \delta_0 \\ \hline w \end{array} \right) \right\|_2^2} = \frac{\left\| \left(\begin{array}{c} \delta_0^2 + w^H w \\ \hline Bw \end{array} \right) \right\|_2^2}{\left\| \left(\begin{array}{c} \delta_0 \\ w \end{array} \right) \right\|_2^2}$$

$$\geq \quad \frac{(\delta_0^2 + w^H w)^2}{\delta_0^2 + w^H w} = \delta_0^2 + w^H w.$$

Thus $\delta_0^2 \geq \delta_0^2 + w^H h$ which means that $w = 0$ and $\tilde{U}^H A \tilde{V} = \left(\begin{array}{c|c} \delta_0 & 0 \\ \hline 0 & B \end{array} \right)$.

By the induction hypothesis, there exists unitary $\check{U} \in \mathbb{C}^{(m-1)\times(m-1)}$, unitary $\check{V} \in \mathbb{C}^{(n-1)\times(n-1)}$, and $\check{D} \in \mathbb{R}^{(m-1)\times(n-1)}$ such that $B = \check{U}\check{D}\check{V}^H$ where $\check{D} = \left(\begin{array}{c|c} \check{D}_{TL} & 0 \\ \hline 0 & 0 \end{array} \right)$ with $\check{D}_{TL} = \mathrm{diag}(\delta_1, \cdots, \delta_{r-1})$.

Now, let

$$U = \tilde{U} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \check{U} \end{array} \right), V = \tilde{V} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \check{V} \end{array} \right), \text{ and } D = \left(\begin{array}{c|c} \delta_0 & 0 \\ \hline 0 & \check{D} \end{array} \right).$$

(There are some really tough to see "checks" in the definition of U, V, and D!!) Then $A = UDV^H$ where U, V, and D have the desired properties.

- **By the Principle of Mathematical Induction** the result holds for all matrices $A \in \mathbb{C}^{m\times n}$ with $m \geq n$.

□

Exercise 13. *Let* $D = \mathrm{diag}(\delta_0, \ldots, \delta_{n-1})$. *Show that* $\|D\|_2 = \max_{i=0}^{n-1} |\delta_i|$.

Exercise 14. *Let* $A = \left(\begin{array}{c} A_T \\ \hline 0 \end{array} \right)$. *Use the SVD of* A *to show that* $\|A\|_2 = \|A_T\|_2$.

Exercise 15. *Assume that* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{n\times n}$ *be unitary matrices. Let* $A, B \in \mathbb{C}^{m\times n}$ *with* $B = UAV^H$. *Show that the singular values of* A *equal the singular values of* B.

Exercise 16. *Let* $A \in \mathbb{C}^{m\times n}$ *with* $A = \left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & B \end{array} \right)$ *and assume that* $\|A\|_2 = \sigma_0$. *Show that* $\|B\|_2 \leq \|A\|_2$. *(Hint: Use the SVD of* B*.)*

Exercise 17. *Prove Lemma 12 for* $m \leq n$.

You can use the following as an outline for your proof: **Proof:** First, let us observe that if $A = 0$ (the zero matrix) then the theorem trivially holds: $A = UDV^H$ where $U = I_{m\times m}$, $V = I_{n\times n}$, and $D = \left(\begin{array}{c|c} & \\ \hline & 0 \end{array} \right)$, so that D_{TL} is 0×0. Thus, w.l.o.g. assume that $A \neq 0$.

We will employ a proof by induction on m.

- **Base case:** $m = 1$. In this case $A = \left(\; \widehat{a}_0^T \; \right)$ where $\widehat{a}_0^T \in \mathbb{R}1 \times n$ is its only row. By assumption, $\widehat{a}_0^T \neq 0$. Then

$$A = \left(\; \widehat{a}_0^T \; \right) = \left(\; 1 \; \right) \left(\|\widehat{a}_0^T\|_2 \right) \left(\; v_0 \; \right)^H$$

where $v_0 = (\widehat{a}_0^T)^H / \|\widehat{a}_0^T\|_2$. Choose $V_1 \in \mathbb{C}^{n\times(n-1)}$ so that $V = \left(\begin{array}{c|c} v_0 & V_1 \end{array} \right)$ is unitary. Then

$$A = \left(\begin{array}{c} \widehat{a}_0^T \end{array} \right) = \left(\begin{array}{c} 1 \end{array} \right) \left(\begin{array}{c|c} \|\widehat{a}_0^T\|_2) & 0 \end{array} \right) \left(\begin{array}{c|c} v_0 & V_1 \end{array} \right)^H = UDV^H$$

where $D_{TL} = \left(\begin{array}{c} \delta_0 \end{array} \right) = \left(\begin{array}{c} \|\widehat{a}_0^T\|_2 \end{array} \right)$ and $U = \left(\begin{array}{c} 1 \end{array} \right)$.

- **Inductive step:** Similarly modify the inductive step of the proof of the theorem.
- **By the Principle of Mathematical Induction** the result holds for all matrices $A \in \mathbb{C}^{m\times n}$ with $m \geq n$.

□

3 *The* Theorem

Theorem 18 (Singular Value Decomposition)**.** *Given $A \in \mathbb{C}^{m\times n}$ there exists unitary $U \in \mathbb{C}^{m\times m}$, unitary $V \in \mathbb{C}^{n\times n}$, and $\Sigma \in \mathbb{R}^{m\times n}$ such that $A = U\Sigma V^H$ where $\Sigma = \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right)$ with $\Sigma_{TL} = \mathrm{diag}(\sigma_0, \cdots, \sigma_{r-1})$ and $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{r-1} > 0$. The $\sigma_0, \ldots, \sigma_{r-1}$ are known as the singular values of A.*

Proof: Notice that the proof of the above theorem is identical to that of Lemma 12. However, thanks to the above exercises, we can conclude that $\|B\|_2 \leq \sigma_0$ in the proof, which then can be used to show that the singular values are found in order. □

Proof:(Alternative) An alternative proof uses Lemma 12 to conclude that $A = UDV^H$. If the entries on the diagonal of D are not ordered from largest to smallest, then this can be fixed by permuting the rows and columns of D, and correspondingly permuting the columns of U and V. □

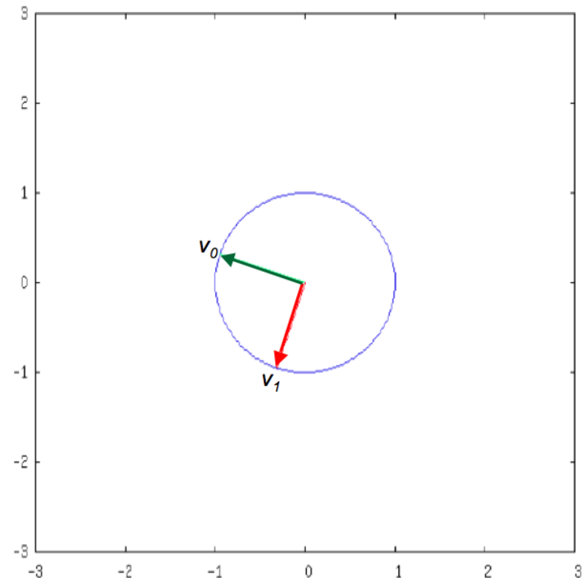
4 Geometric Interpretation (Again)

We will now quickly illustrate what the SVD Theorem tells us about matrix-vector multiplication (linear transformations) by examining the case where $A \in \mathbb{R}^{2\times 2}$. Let $A = U\Sigma V^T$ be its SVD decomposition. (Notice that all matrices are now real valued, and hence $V^H = V^T$.) Partition

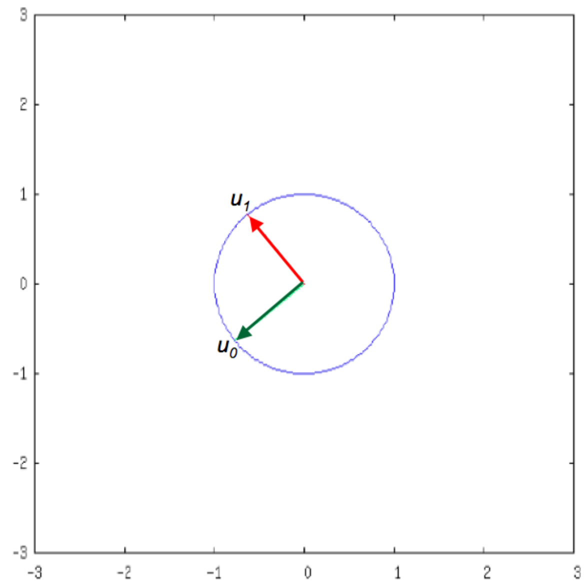
$$A = \left(\begin{array}{c|c} u_0 & u_1 \end{array} \right) \left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & \sigma_1 \end{array} \right) \left(\begin{array}{c|c} v_0 & v_1 \end{array} \right)^T.$$

Since U and V are unitary matrices, $\{u_0, u_1\}$ and $\{v_0, v_1\}$ form orthonormal bases for the range and domain of A, respectively:

$\mathbb{R}^2$: Domain of A.



$\mathbb{R}^2$: Range (codomain) of A.



Let us manipulate the decomposition a little:

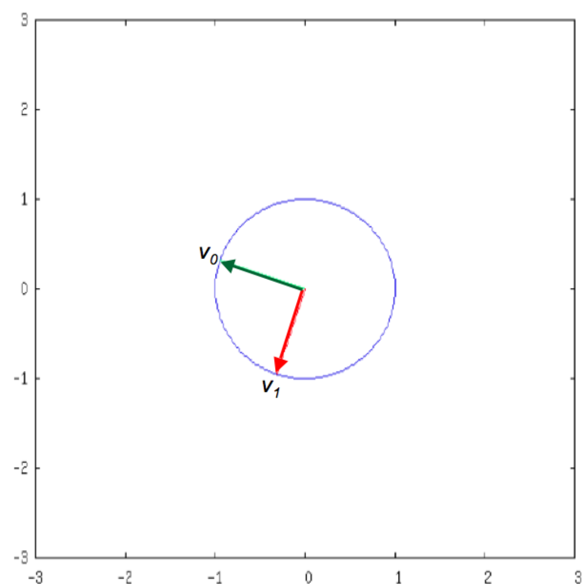
$$\begin{aligned} A &= \left(\begin{array}{c|c} u_0 & u_1 \end{array} \right) \left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & \sigma_1 \end{array} \right) \left(\begin{array}{c|c} v_0 & v_1 \end{array} \right)^T = \left[\left(\begin{array}{c|c} u_0 & u_1 \end{array} \right) \left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & \sigma_1 \end{array} \right) \right] \left(\begin{array}{c|c} v_0 & v_1 \end{array} \right)^T \\ &= \left(\begin{array}{c|c} \sigma_0 u_0 & \sigma_1 u_1 \end{array} \right) \left(\begin{array}{c|c} v_0 & v_1 \end{array} \right)^T. \end{aligned}$$

Now let us look at how A transforms v_0 and v_1:

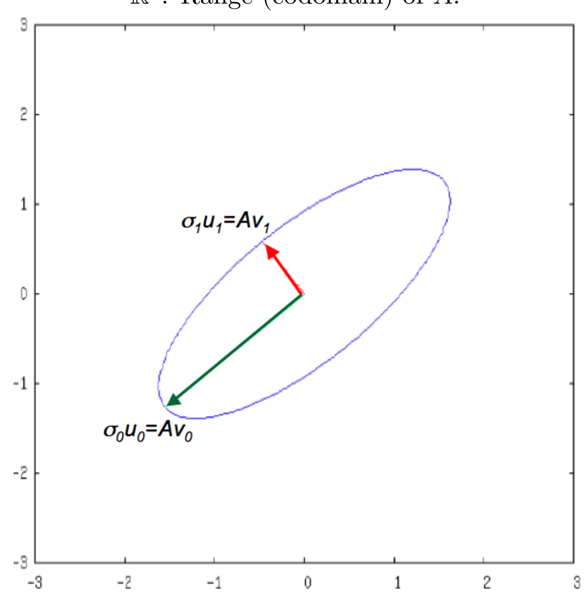
$$Av_0 = \left(\begin{array}{c|c} \sigma_0 u_0 & \sigma_1 u_1 \end{array} \right) \left(\begin{array}{c|c} v_0 & v_1 \end{array} \right)^T v_0 = \left(\begin{array}{c|c} \sigma_0 u_0 & \sigma_1 u_1 \end{array} \right) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) = \sigma_0 u_0$$

and similarly $Av_1 = \sigma_1 u_1$. This motivates the pictures

$\mathbb{R}^2$: Domain of A.



$\mathbb{R}^2$: Range (codomain) of A.



Now let us look at how A transforms any vector with (Euclidean) unit length. Notice that $x = \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}$ means that

$$x = \chi_0 e_0 + \chi_1 e_1,$$

where e_0 and e_1 are the unit basis vectors. Thus, χ_0 and χ_1 are the coefficients when x is expressed using e_0 and e_1 as basis. However, we can also express x in the basis given by v_0 and v_1:

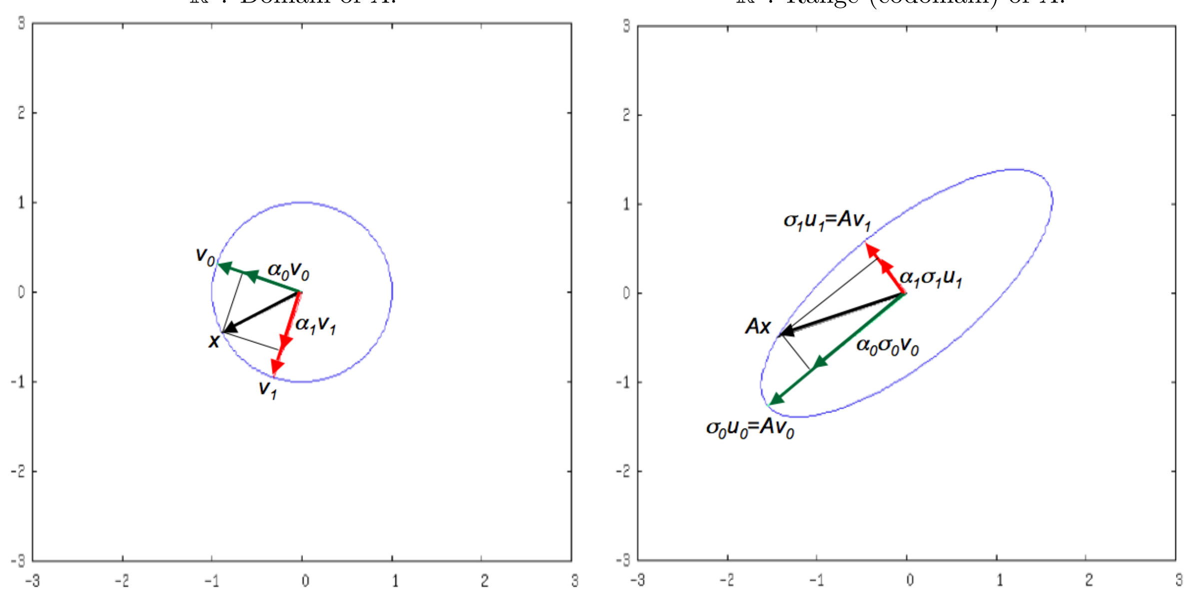
$$\begin{aligned} x &= \underbrace{VV^T}_{I} x = \left(v_0 \mid v_1 \right) \left(v_0 \mid v_1 \right)^T x = \left(v_0 \mid v_1 \right) \begin{pmatrix} v_0^T x \\ \hline v_1^T x \end{pmatrix} \\ &= \underbrace{v_0^T x}_{\alpha_0} v_0 + \underbrace{v_1^T x}_{\alpha_1} v_1 = \alpha_0 v_0 + \alpha_0 v_1 = \left(v_0 \mid v_1 \right) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}. \end{aligned}$$

Thus, in the basis formed by v_0 and v_1, its coefficients are α_0 and α_1. Now,

$$\begin{aligned} Ax &= \left(\sigma_0 u_0 \mid \sigma_1 u_1 \right) \left(v_0 \mid v_1 \right)^T x = \left(\sigma_0 u_0 \mid \sigma_1 u_1 \right) \left(v_0 \mid v_1 \right)^T \left(v_0 \mid v_1 \right) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \\ &= \left(\sigma_0 u_0 \mid \sigma_1 u_1 \right) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \alpha_0 \sigma_0 u_0 + \alpha_1 \sigma_1 u_1. \end{aligned}$$

This is illustrated by the following picture, which also captures the fact that the unit ball is mapped to an "ellipse"[1] with major axis equal to $\sigma_0 = \|A\|_2$ and minor axis equal to σ_1:

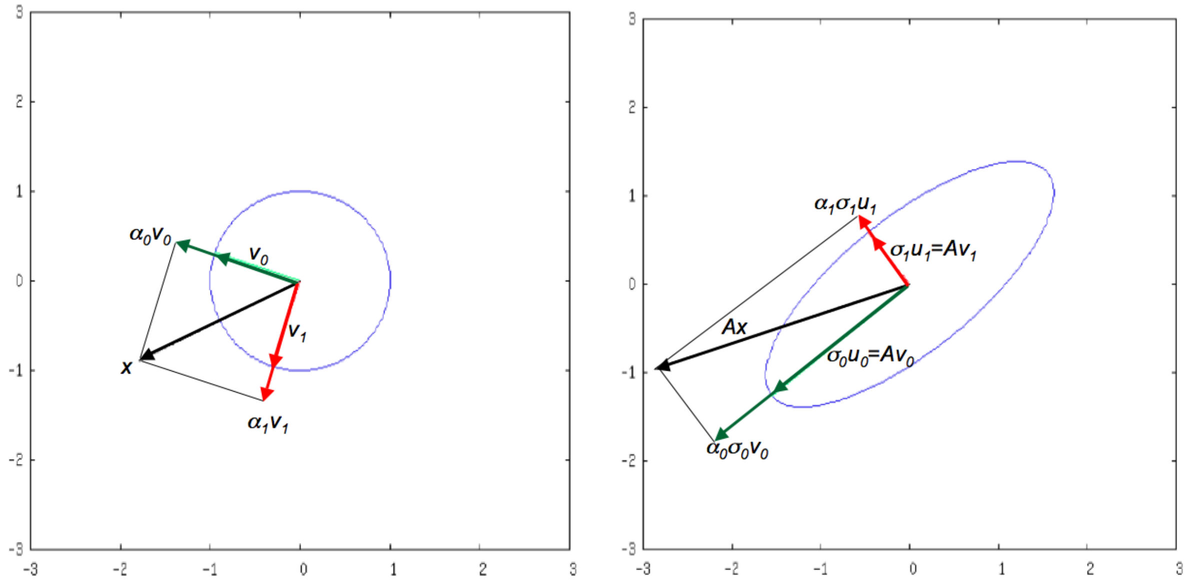
$\mathbb{R}^2$: Domain of A. $\mathbb{R}^2$: Range (codomain) of A.



Finally, we show the same insights for general vector x (not necessarily of unit length).

[1] It is not clear that it is actually an ellipse and this is not important to our observations.

$\mathbb{R}^2$: Domain of A. $\mathbb{R}^2$: Range (codomain) of A.



Another observation is that *if* one picks the right basis for the domain and codomain, then the computation Ax simplifies to a matrix multiplication with a diagonal matrix. Let us again illustrate this for nonsingular $A \in \mathbb{R}^{2\times 2}$ with

$$A = \underbrace{\left(\begin{array}{c|c} u_0 & u_1 \end{array}\right)}_{U} \underbrace{\left(\begin{array}{c|c} \sigma_0 & 0 \\ \hline 0 & \sigma_1 \end{array}\right)}_{\Sigma} \underbrace{\left(\begin{array}{c|c} v_0 & v_1 \end{array}\right)}_{V}^{T}.$$

Now, if we chose to express y using u_0 and u_1 as the basis and express x using v_0 and v_1 as the basis, then

$$\begin{array}{rcl} \widehat{y} & = & \underbrace{UU^T}_{I} y = (u_0^T y) u_0 + (u_1^T y) u_1 = \left(\begin{array}{c} \widehat{\psi}_0 \\ \hline \widehat{\psi}_1 \end{array}\right) \\ \widehat{x} & = & \underbrace{VV^T}_{I} x = (v_0^T x) v_0 + (v_1^T x) v_1 == \left(\begin{array}{c} \widehat{\chi}_0 \\ \hline \widehat{\chi}_1. \end{array}\right). \end{array}$$

If $y = Ax$ then

$$U \underbrace{U^T y}_{\widehat{y}} = \underbrace{U \Sigma V^T x}_{Ax} = U \Sigma \widehat{x}$$

so that $\widehat{y} = \Sigma \widehat{x}$ and

$$\left(\begin{array}{c} \widehat{\psi}_0 \\ \hline \widehat{\psi}_1. \end{array}\right) = \left(\begin{array}{c} \sigma_0 \widehat{\chi}_0 \\ \hline \sigma_1 \widehat{\chi}_1. \end{array}\right).$$

These observation generalize to $A \in \mathbb{C}^{m\times m}$.

5 Consequences of the SVD Theorem

Throughout this section we will assume that

- $A = U\Sigma V^H$ is the SVD of $A \in \mathbb{C}^{m\times n}$, with U and V unitary and Σ diagonal.
- $\Sigma = \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array}\right)$ where $\Sigma_{TL} = \mathrm{diag}(\sigma_0, \ldots, \sigma_{r-1})$ with $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{r-1} > 0$.
- $U = \left(\begin{array}{c|c} U_L & U_R \end{array}\right)$ with $U_L \in \mathbb{C}^{m\times r}$.
- $V = \left(\begin{array}{c|c} V_L & V_R \end{array}\right)$ with $V_L \in \mathbb{C}^{n\times r}$.

We first generalize the observations we made for $A \in \mathbb{R}^{2\times 2}$. Let us track what the effect of $Ax = U\Sigma V^H x$ is on vector x. We assume that $m \geq n$.

- Let $U = \left(\begin{array}{c|c|c} u_0 & \cdots & u_{m-1} \end{array}\right)$ and $V = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array}\right)$.
- Let

$$
\begin{array}{rcl}
x & = & VV^H x = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array}\right)\left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array}\right)^H x = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array}\right)\left(\begin{array}{c} v_0^H x \\ \hline \vdots \\ \hline v_{n-1}^H x \end{array}\right) \\
& = & v_0^H x v_0 + \cdots + v_{n-1}^H x v_{n-1}.
\end{array}
$$

  This can be interpreted as follows: vector x can be written in terms of the usual basis of $\mathbb{C}^n$ as $\chi_0 e_0 + \cdots + \chi_1 e_{n-1}$ or in the orthonormal basis formed by the columns of V as $v_0^H x v_0 + \cdots + v_{n-1}^H x v_{n-1}$.
- Notice that $Ax = A(v_0^H x v_0 + \cdots + v_{n-1}^H x v_{n-1}) = v_0^H x A v_0 + \cdots + v_{n-1}^H x A v_{n-1}$ so that we next look at how A transforms each v_i: $Av_i = U\Sigma V^H v_i = U\Sigma e_i = \sigma_i U e_i = \sigma_i u_i$.
- Thus, another way of looking at Ax is

$$
\begin{array}{rcl}
Ax & = & v_0^H x A v_0 + \cdots + v_{n-1}^H x A v_{n-1} \\
& = & v_0^H x \sigma_0 u_0 + \cdots + v_{n-1}^H x \sigma_{n-1} u_{n-1} \\
& = & \sigma_0 u_0 v_0^H x + \cdots + \sigma_{n-1} u_{n-1} v_{n-1}^H x \\
& = & \left(\sigma_0 u_0 v_0^H + \cdots + \sigma_{n-1} u_{n-1} v_{n-1}^H\right) x.
\end{array}
$$

Corollary 19. *$A = U_L \Sigma_{TL} V_L^H$. This is called the reduced SVD of A.*

Proof:

$$
A = U\Sigma V^H = \left(\begin{array}{c|c} U_L & U_R \end{array}\right)\left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} V_L & V_R \end{array}\right)^H = U_L \Sigma_{TL} V_L^H.
$$

□

Corollary 20. *Let $A = U_L \Sigma_{TL} V_L^H$ be the reduced SVD with $U_L = \left(\begin{array}{c|c|c} u_0 & \cdots & u_{r-1} \end{array}\right)$, $\Sigma_{TL} = \mathrm{diag}(\sigma_0, \ldots, \sigma_{r-1})$, and $V_L = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{r-1} \end{array}\right)$. Then $A = \sigma_0 u_0 v_0^H + \sigma_1 u_1 v_1^H + \cdots + \sigma_{r-1} u_{r-1} v_{r-1}^H$ (each term nonzero and an outer product, and hence a rank-1 matrix).*

Proof: We leave the proof as an exercise. □

Corollary 21. *$\mathcal{C}(A) = \mathcal{C}(U_L)$.*

Proof:

- Let $y \in \mathcal{C}(A)$. Then there exists $x \in \mathbb{C}^n$ such that $y = Ax$ (by the definition of $y \in \mathcal{C}(A)$). But then

$$y = Ax = U_L \underbrace{\Sigma_{TL} V_L^H x}_{z} = U_L z,$$

i.e., there exists $z \in \mathbb{C}^r$ such that $y = U_L z$. This means $y \in \mathcal{C}(U_L)$.

- Let $y \in \mathcal{C}(U_L)$. Then there exists $z \in \mathbb{C}^r$ such that $y = U_L z$. But then

$$y = U_L z = U_L \underbrace{\Sigma_{TL} \Sigma_{TL}^{-1}}_{I} z = U_L \Sigma_{TL} \underbrace{V_L^H V_L}_{I} \Sigma_{TL}^{-1} z = A \underbrace{V_L \Sigma_{TL}^{-1} z}_{x} = Ax$$

so that there exists $x \in \mathbb{C}^n$ such that $y = Ax$, i.e., $y \in \mathcal{C}(A)$.

□

Corollary 22. *The rank of A is r.*

Proof: The rank of A equals the dimension of $\mathcal{C}(A) = \mathcal{C}(U_L)$. But the dimension of $\mathcal{C}(U_L)$ is clearly r. □

Corollary 23. $\mathcal{N}(A) = \mathcal{C}(V_R)$.

Proof:

- Let $x \in \mathcal{N}(A)$. Then

$$\begin{aligned} x &= \underbrace{VV^H}_{I} x = \left(V_L \mid V_R \right) \left(V_L \mid V_R \right)^H x = \left(V_L \mid V_R \right) \left(\frac{V_L^H}{V_R^H} \right) x \\ &= \left(V_L \mid V_R \right) \left(\frac{V_L^H x}{V_R^H x} \right) = V_L V_L^H x + V_R V_R^H x. \end{aligned}$$

If we can show that $V_L^H x = 0$ then $x = V_R z$ where $z = V_R^H x$. Assume that $V_L^H x \neq 0$. Then $\Sigma_{TL}(V_L^H x) \neq 0$ (since Σ_{TL} is nonsingular) and $U_L(\Sigma_{TL}(V_L^H x)) \neq 0$ (since U_L has linearly independent columns). But that contradicts the fact that $Ax = U_L \Sigma_{TL} V_L^H x = 0$.

- Let $x \in \mathcal{C}(V_R)$. Then $x = V_R z$ for some $z \in \mathbb{C}^r$ and $Ax = U_L \Sigma_{TL} \underbrace{V_L^H V_R}_{0} z = 0$.

□

Corollary 24. *For all $x \in \mathbb{C}^n$ there exists $z \in \mathcal{C}(V_L)$ such that $Ax = Az$.*

Proof:

$$\begin{aligned} Ax &= A \underbrace{VV^H}_{I} x = A \left(V_L \mid V_R \right) \left(V_L \mid V_R \right)^H x \\ &= A \left(V_L V_L^H x + V_R V_R^H x \right) = A V_L V_L^H x + A V_R V_R^H x \\ &= A V_L V_L^H x + U_L \Sigma_{TL} \underbrace{V_L^H V_R}_{0} V_R^H x = A \underbrace{V_L V_L^H x}_{z}. \end{aligned}$$

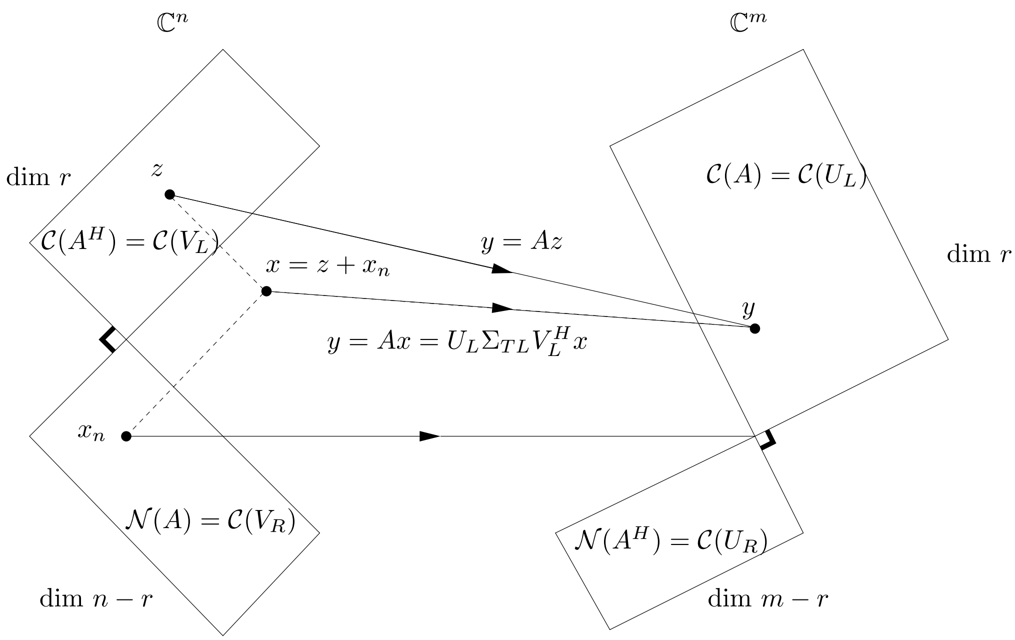


Figure 1: A pictorial description of how $x = z + x_n$ is transformed by $A \in \mathbb{C}^{m \times n}$ into $y = Ax = A(z + x_n)$. We see that $\mathcal{C}(V_L)$ and $\mathcal{C}(V_R)$ are orthogonal complements of each other within $\mathbb{C}^n$. Similarly, $\mathcal{C}(U_L)$ and $\mathcal{C}(U_R)$ are orthogonal complements of each other within $\mathbb{C}^m$. Any vector x can be written as the sum of a vector $z \in \mathcal{C}(V_R)$ and $x_n \in \mathcal{C}(V_C) = \mathcal{N}(A)$.

Alternative proof (which uses the last corollary):

$$Ax = A\left(V_L V_L^H x + V_R V_R^H x\right) = A V_L V_L^H x + A \underbrace{V_R V_R^H x}_{\in \mathcal{N}(A)} = A \underbrace{V_L V_L^H x}_{z}.$$

□

The proof of the last corollary also shows that

Corollary 25. *Any vector $x \in \mathbb{C}^n$ can be written as $x = z + x_n$ where $z \in \mathcal{C}(V_L)$ and $x_n \in \mathcal{N}(A) = \mathcal{C}(V_R)$.*

Corollary 26. *$A^H = V_L \Sigma_{TL} U_L^H$ so that $\mathcal{C}(A^H) = \mathcal{C}(V_L)$ and $\mathcal{N}(A^H) = \mathcal{C}(U_R)$.*

The above corollaries are summarized in Figure 1.

Theorem 27. *Let $A \in \mathbb{C}^{n \times n}$ be nonsingular. Let $A = U \Sigma V^H$ be its SVD. Then*

1. *The SVD is the reduced SVD.*
2. *$\sigma_{n-1} \neq 0$.*
3. *If*
$$U = \left(\begin{array}{c|c|c} u_0 & \cdots & u_{n-1} \end{array} \right), \Sigma = \mathrm{diag}(\sigma_0, \ldots, \sigma_{n-1}), \text{ and } V = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array} \right),$$
then
$$A^{-1} = (VP^T)(P\Sigma^{-1}P^T)(UP^T)^H = \left(\begin{array}{c|c|c} v_{n-1} & \cdots & v_0 \end{array} \right) \mathrm{diag}(\frac{1}{\sigma_{n-1}}, \ldots, \frac{1}{\sigma_0}) \left(\begin{array}{c|c|c} u_{n-1} & \cdots & u_0 \end{array} \right),$$

where $P = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$ *is the permutation matrix such that* Px *reverses the order of the entries in* x. *(Note: for this permutation matrix,* $P^T = P$. *In general, this is not the case. What is the case for all permutation matrices* P *is that* $P^T P = PP^T = I$.*)*

4. $\|A^{-1}\|_2 = 1/\sigma_{n-1}$.

Proof: The only item that is less than totally obvious is (3). Clearly $A^{-1} = V\Sigma^{-1}U^H$. The problem is that in Σ^{-1} the diagonal entries are not ordered from largest to smallest. The permutation fixes this. □

Corollary 28. *If* $A \in \mathbb{C}^{m\times n}$ *has linearly independent columns then* $A^H A$ *is invertible (nonsingular) and* $(A^H A)^{-1} = V_L \left(\Sigma_{TL}^2\right)^{-1} V_L^H$.

Proof: Since A has linearly independent columns, $A = U_L \Sigma_{TL} V_L^H$ is the reduced SVD where U_L has n columns and V_L is unitary. Hence

$$A^H A = (U_L\Sigma_{TL}V_L^H)^H U_L\Sigma_{TL}V_L^H = V_L\Sigma_{TL}^H U_L^H U_L \Sigma_{TL} V_L^H = V_L \Sigma_{TL}\Sigma_{TL}V_L^H = V_L\Sigma_{TL}^2 V_L^H.$$

Since V_L is unitary and Σ_{TL} is diagonal with nonzero diagonal entries, tey are both nonsingular. Thus

$$\left(V_L\Sigma_{TL}^2 V_L^H\right)\left(V_L\left(\Sigma_{TL}^2\right)^{-1}V_L^H)\right) = I.$$

This means $A^T A$ is invertible and $(A^T A)^{-1}$ is as given. □

6 Projection onto the Column Space

Definition 29. *Let* $U_L \in \mathbb{C}^{m\times k}$ *have orthonormal columns. The projection of a vector* $y \in \mathbb{C}^m$ *onto* $\mathcal{C}(U_L)$ *is the vector* $U_L x$ *that minimizes* $\|y - U_L x\|_2$, *where* $x \in \mathbb{C}^k$. *We will also call this vector* y *the component of* x *in* $\mathcal{C}(U_L)$.

Theorem 30. *Let* $U_L \in \mathbb{C}^{m\times k}$ *have orthonormal columns. The projection of* y *onto* $\mathcal{C}(U_L)$ *is given by* $U_L U_L^H y$.

Proof: The vector $U_L x$ that we want must satisfy

$$\|U_L x - y\|_2 = \min_{w\in\mathbb{C}^k} \|U_L w - y\|_2.$$

Now, the 2-norm is invariant under multiplication by the unitary matrix $U^H = \left(\begin{array}{c|c} U_L & U_R \end{array}\right)^H$

$$\begin{aligned}
\|U_L x - y\|_2^2 &= \min_{w\in\mathbb{C}^k} \|U_L w - y\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left\|U^H(U_L w - y)\right\|_2^2 \quad \text{(since the two norm is preserved)} \\
&= \min_{w\in\mathbb{C}^k} \left\|\left(\begin{array}{c|c} U_L & U_R \end{array}\right)^H (U_L w - y)\right\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left\|\left(\begin{array}{c} U_L^H \\ \hline U_R^H \end{array}\right)(U_L w - y)\right\|_2^2
\end{aligned}$$

$$
\begin{aligned}
&= \min_{w\in\mathbb{C}^k} \left\| \left(\begin{array}{c} U_L^H \\ \hline U_R^H \end{array} \right) U_L w - \left(\begin{array}{c} U_L^H \\ \hline U_R^H \end{array} \right) y \right\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left\| \left(\begin{array}{c} U_L^H U_L w \\ \hline U_R^H U_L w \end{array} \right) - \left(\begin{array}{c} U_L^H y \\ \hline U_R^H y \end{array} \right) \right\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left\| \left(\begin{array}{c} w \\ \hline 0 \end{array} \right) - \left(\begin{array}{c} U_L^H y \\ \hline U_R^H y \end{array} \right) \right\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left\| \left(\begin{array}{c} w - U_L^H y \\ \hline -U_R^H y \end{array} \right) \right\|_2^2 \\
&= \min_{w\in\mathbb{C}^k} \left(\left\| w - U_L^H y \right\|_2^2 + \left\| -U_R^H y \right\|_2^2 \right) \quad \left(\text{since } \left\| \left(\begin{array}{c} u \\ \hline v \end{array} \right) \right\|_2^2 = \|u\|_2^2 + \|v\|_2^2 \right) \\
&= \left(\min_{w\in\mathbb{C}^k} \left\| w - U_L^H y \right\|_2^2 \right) + \left\| U_R^H y \right\|_2^2 .
\end{aligned}
$$

This is minimized when $w = U_L^H y$. Thus, the vector that is closest to y in the space spanned by U_L is given by $x = U_L U_L^H y$. □

Corollary 31. *Let $A \in \mathbb{C}^{m\times n}$ and $A = U_L \Sigma_{TL} V_L^H$ be its reduced SVD. Then the projection of $y \in \mathbb{C}^m$ onto $\mathcal{C}(A)$ is given by $U_L U_L^H y$.*

Proof: This follows immediately from the fact that $\mathcal{C}(A) = \mathcal{C}(U_L)$. □

Corollary 32. *Let $A \in \mathbb{C}^{m\times n}$ have linearly independent columns. Then the projection of $y \in \mathbb{C}^m$ onto $\mathcal{C}(A)$ is given by $A(A^H A)^{-1} A^H y$.*

Proof: From Corrolary 28, we know that $A^H A$ is nonsingular and that $(A^H A)^{-1} = V_L \left(\Sigma_{TL}^2\right)^{-1} V_L^H$. Now,

$$
\begin{aligned}
A(A^H A)^{-1} A^H y &= (U_L \Sigma_{TL} V_L^H)(V_L \left(\Sigma_{TL}^2\right)^{-1} V_L^H)(U_L \Sigma_{TL} V_L^H)^H y \\
&= U_L \Sigma_{TL} \underbrace{V_L^H V_L}_{I} \Sigma_{TL}^{-1} \Sigma_{TL}^{-1} \underbrace{V_L^H V_L}_{I} \Sigma_{TL} U_L^H y = U_L U_L^H y.
\end{aligned}
$$

Hence the projection of y onto $\mathcal{C}(A)$ is given by $A(A^H A)^{-1} A^H y$. □

Definition 33. *Let A have linearly independent columns. Then $(A^H A)^{-1} A^H$ is called the pseudo-inverse or Moore-Penrose generalized inverse of matrix A.*

7 Low-rank Approximation of a Matrix

Theorem 34. *Let $A \in \mathbb{C}^{m\times n}$ have SVD $A = U \Sigma V^H$ and assume A has rank r. Partition*

$$
U = \left(\begin{array}{c|c} U_L & U_R \end{array} \right), \quad V = \left(\begin{array}{c|c} V_L & V_R \end{array} \right), \quad \textit{and} \quad \Sigma = \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right),
$$

where $U_L \in \mathbb{C}^{m\times k}$*,* $V_L \in \mathbb{C}^{n\times k}$*, and* $\Sigma_{TL} \in \mathbb{R}^{k\times k}$ *with* $k \le r$*. Then* $B = U_L \Sigma_{TL} V_L^H$ *is the matrix in* $\mathbb{C}^{m\times n}$ *closest to* A *in the following sense:*

$$\|A - B\|_2 = \min_{\substack{C \in \mathbb{C}^{m\times n} \\ \mathrm{rank}(C) \le k}} \|A - C\|_2 = \sigma_k.$$

Proof: First, if B is as defined, then clearly $\|A - B\|_2 = \sigma_k$:

$$\begin{aligned}
\|A - B\|_2 &= \|U^H(A-B)V\|_2 = \|U^H AV - U^H BV\|_2 \\
&= \left\| \Sigma - \left(\begin{array}{c|c} U_L & U_R \end{array} \right)^H B \left(\begin{array}{c|c} V_L & V_R \end{array} \right) \right\|_2 = \left\| \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right) - \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & 0 \end{array} \right) \right\|_2 \\
&= \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right) \right\|_2 = \|\Sigma_{BR}\|_2 = \sigma_k
\end{aligned}$$

Next, assume that C has rank $t \le k$ and $\|A - C\|_2 < \|A - B\|_2$. We will show that this leads to a contradiction.

- The null space of C has dimension at least $n - k$ since $\dim(\mathcal{N}(C)) + \mathrm{rank}(C) = n$.
- If $x \in \mathcal{N}(C)$ then
$$\|Ax\|_2 = \|(A - C)x\|_2 \le \|A - C\|_2 \|x\|_2 < \sigma_k \|x\|_2.$$
- Partition $U = \left(\begin{array}{c|c|c} u_0 & \cdots & u_{m-1} \end{array} \right)$ and $V = \left(\begin{array}{c|c|c} v_0 & \cdots & v_{n-1} \end{array} \right)$. Then $\|Av_j\|_2 = \|\sigma_j u_j\|_2 = \sigma_j \ge \sigma_s$ for $j = 0, \ldots, k$. Now, let x be any linear combination of $v_0, \ldots, v_k$: $x = \alpha_0 v_0 + \cdots + \alpha_k v_k$. Notice that
$$\|x\|_2^2 = \|\alpha_0 v_0 + \cdots + \alpha_k v_k\|_2^2 \le |\alpha_0|^2 + \cdots |\alpha_k|^2.$$
Then
$$\begin{aligned}
\|Ax\|_2^2 &= \|A(\alpha_0 v_0 + \cdots + \alpha_k v_k)\|_2^2 = \|\alpha_0 A v_0 + \cdots + \alpha_k A v_k\|_2^2 \\
&= \|\alpha_0 \sigma_0 u_0 + \cdots + \alpha_k \sigma_k u_k\|_2^2 = \|\alpha_0 \sigma_0 u_0\|_2^2 + \cdots + \|\alpha_k \sigma_k u_k\|_2^2 \\
&= |\alpha_0|^2 \sigma_0^2 + \cdots + |\alpha_k|^2 \sigma_k^2 \ge (|\alpha_0|^2 + \cdots + |\alpha_k|^2)\sigma_k^2
\end{aligned}$$
so that $\|Ax\|_2 \ge \sigma_k \|x\|_2$. In other words, vectors in the subspace of all linear combinations of $\{v_0, \ldots, v_k\}$ satisfy $\|Ax\|_2 \ge \sigma_k \|x\|_2$. The dimension of this subspace is $k + 1$ (since $\{v_0, \cdots, v_k\}$ form an orthonormal basis).
- Both these subspaces are subspaces of $\mathbb{C}^n$. Since their dimensions add up to more than n there must be at least one nonzero vector z that satisfies both $\|Az\|_2 < \sigma_k \|z\|_2$ and $\|Az\|_2 \ge \sigma_k \|z\|_2$, which is a contradiction.

□

The above theorem tells us how to pick the best approximation with given rank to a given matrix.

8 An Application

Let $Y \in \mathbb{R}^{m\times n}$ be a matrix that, for example, stores a picture. In this case, the (i, j) entry in Y is, for example, a number that represents the grayscale value of pixel (i, j). The following instructions, executed in octave or matlab, generate the picture of Mexican artist Frida Kahlo in Figure 2(top-left). The file `FridaPNG.png` can be found at `http://www.cs.utexas.edu/users/flame/Notes/FridaPNG.png`.

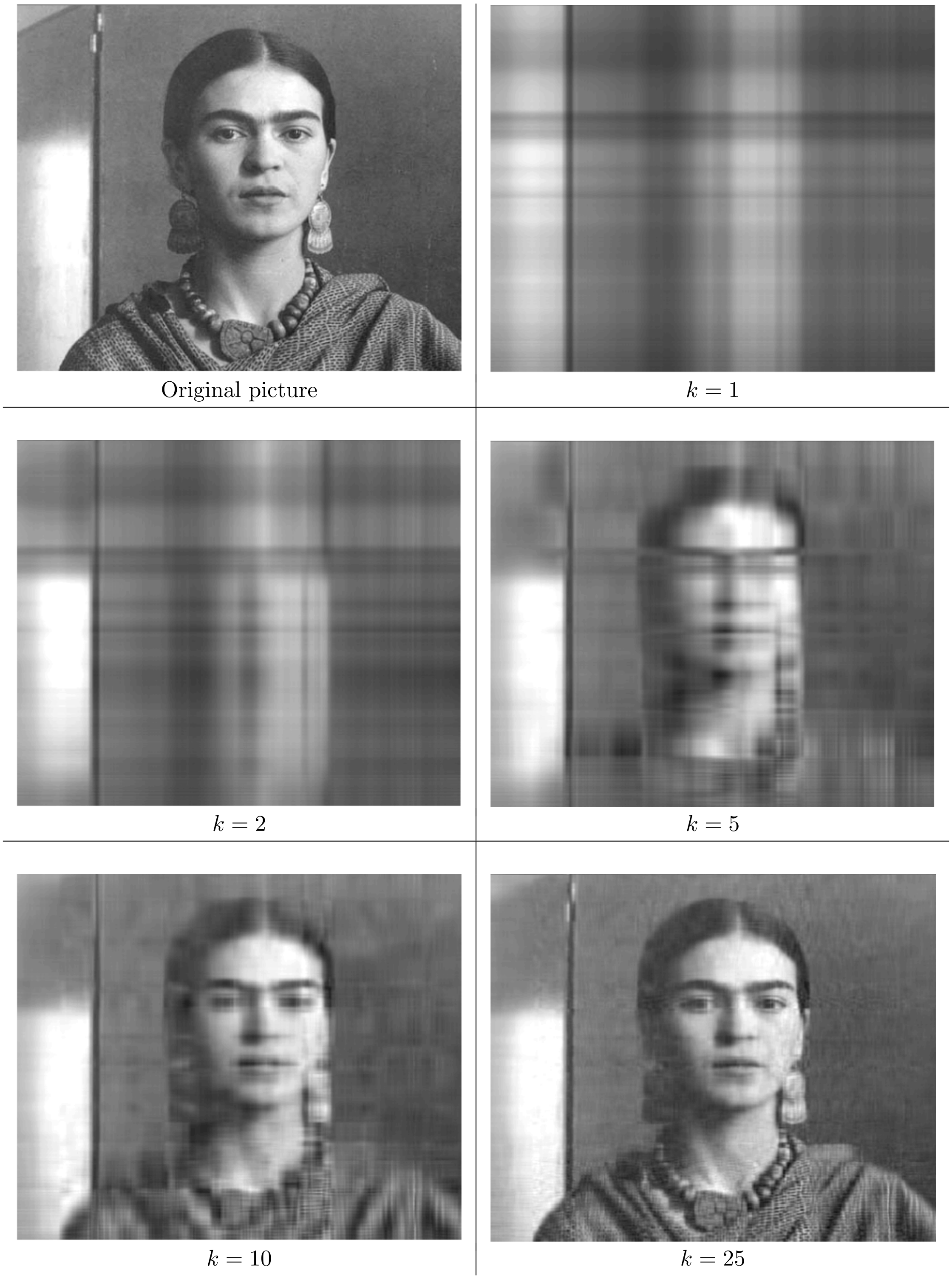


Figure 2: Multiple pictures as generated by the code

```
octave> IMG = imread( 'FridaPNG.png' ); % this reads the image
octave> Y = IMG( :,:,1 );
octave> imshow( Y )                    % this dispays the image
```

Although the picture is black and white, it was read as if it is a color image, which means a $m \times n \times 3$ array of pixel information is stored. Setting `Y = IMG( :,:,1 )` extracts a single matrix of pixel information. (If you start with a color picture, you will want to approximate `IMG( :,:,1)`, `IMG( :,:,2)`, and `IMG( :,:,3)` separately.)

Now, let $Y = U\Sigma V^T$ be the SVD of matrix Y. Partition, conformally,

$$U = \left(U_L \,\middle|\, U_R \right), \quad V = \left(V_L \,\middle|\, V_R \right), \quad \text{and} \quad \Sigma = \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right),$$

where U_L and V_L have k columns and Σ_{TL} is $k \times k$. so that

$$\begin{aligned} Y &= \left(U_L \,\middle|\, U_R \right) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right) \left(V_L \,\middle|\, V_R \right)^T \\ &= \left(U_L \,\middle|\, U_R \right) \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array} \right) \left(\begin{array}{c} V_L^T \\ \hline V_R^T \end{array} \right) \\ &= \left(U_L \,\middle|\, U_R \right) \left(\begin{array}{c} \Sigma_{TL} V_L^T \\ \hline \Sigma_{BR} V_R^T \end{array} \right) \\ &= U_L \Sigma_{TL} V_L^T + U_R \Sigma_{BR} V_R^T. \end{aligned}$$

Recall that then $U_L \Sigma_{TL} V_L^T$ is the best rank-k approximation to Y.

Let us approximate the matrix that stores the picture with $U_L \Sigma_{TL} V_L^T$:

```
>> IMG = imread( 'FridaPNG.png' );  % read the picture
>> Y = IMG( :,:,1 );
>> imshow( Y ); % this dispays the image
>> k = 1;
>> [ U, Sigma, V ] = svd( Y );
>> UL = U( :, 1:k );            % first k columns
>> VL = V( :, 1:k );            % first k columns
>> SigmaTL = Sigma( 1:k, 1:k ); % TL submatrix of Sigma
>> Yapprox = uint8( UL * SigmaTL * VL' );
>> imshow( Yapprox );
```

As one increases k, the approximation gets better, as illustrated in Figure 2. The graph in Figure 3 helps explain. The original matrix Y is 387×469, with $181,503$ entries. When $k = 10$, matrices U, V, and Σ are 387×10, 469×10 and 10×10, respectively, requiring only $8,660$ entries to be stores.

9 SVD and the Condition Number of a Matrix

In "Notes on Norms" we saw that if $Ax = b$ and $A(x + \delta x) = b + \delta b$, then

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \kappa_2(A) \frac{\|\delta b\|_2}{\|b\|_2},$$

where $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$ is the condition number of A, using the 2-norm.

Exercise 35. *Show that if $A \in \mathbb{C}^{m \times m}$ is nonsingular, then*

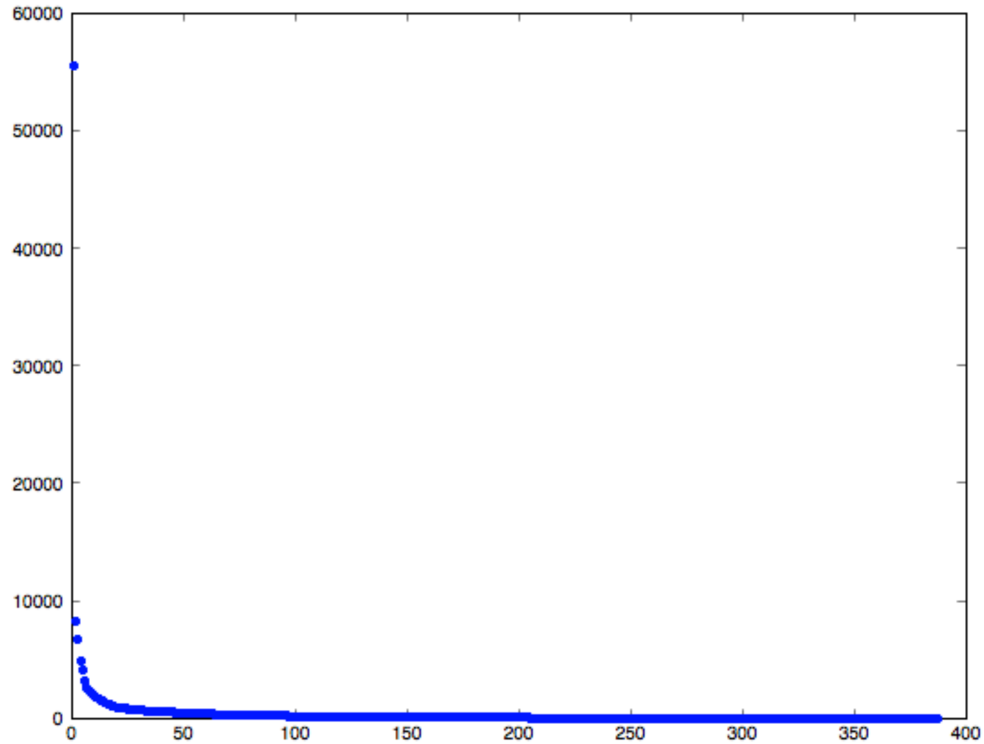
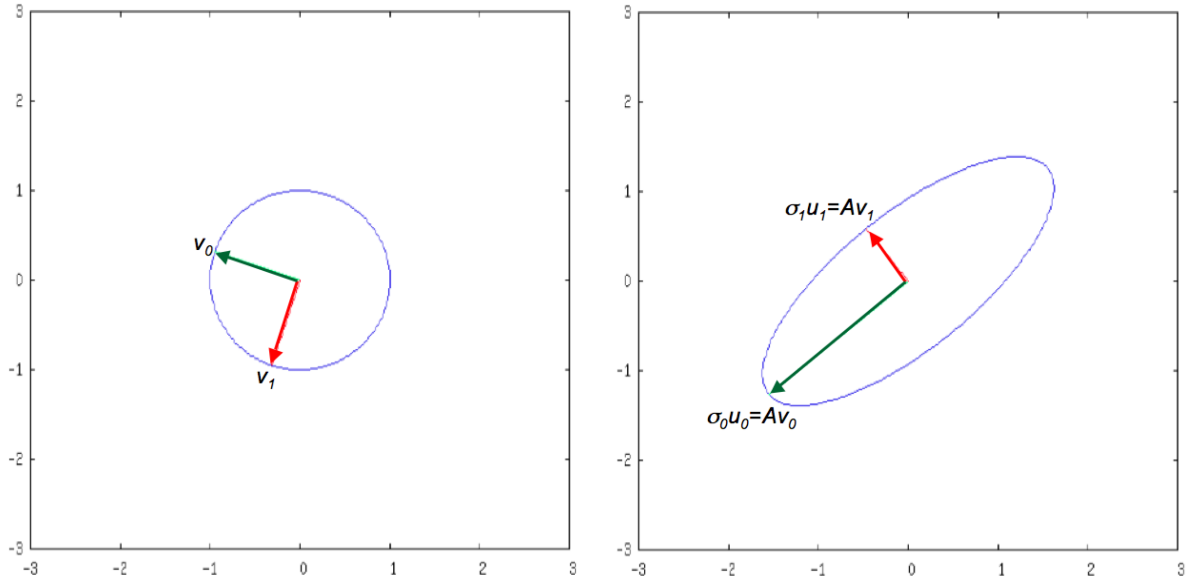


Figure 3: Distribution of singular values for the picture.

- $\|A\|_2 = \sigma_0$, *the largest singular value;*
- $\|A^{-1}\|_2 = 1/\sigma_{m-1}$, *the inverse of the smallest singular value; and*
- $\kappa_2(A) = \sigma_0/\sigma_{m-1}$.

If we go back to the example of $A \in \mathbb{R}^{2\times 2}$, recall the following pictures that shows how A transforms the unit circle:

$\mathbb{R}^2$: Domain of A. $\mathbb{R}^2$: Range (codomain) of A.



In this case, the ratio σ_0/σ_{n-1} represents the ratio between the major and minor axes of the "ellipse" on the right.

10 An Algorithm for Computing the SVD?

It would seem that the proof of the existence of the SVD is constructive in the sense that it provides an algorithm for computing the SVD of a given matrix $A \in \mathbb{C}^{m\times m}$. Not so fast! Observe that

- Computing $\|A\|_2$ is nontrivial.
- Computing the vector that maximizes $\max_{\|x\|_2=1} \|Ax\|_2$ is nontrivial.
- Given a vector q_0 computing vectors $q_0, \ldots, q_{m-1}$ is expensive (as we will see when we discuss the QR factorization).

Towards the end of the course we will discuss algorithms for computing the eigenvalues and eigenvectors of a matrix, and related algorithms for computing the SVD.