# Parametric surfaces

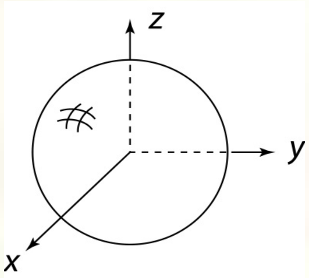
# Reading

- Required:
  - Watt, 2.1.4, 3.4-3.5.

- Optional
  - Watt, 3.6.
  - Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling,* 1987.

# Mathematical surface representations

- Explicit $z = f(x,y)$ (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?



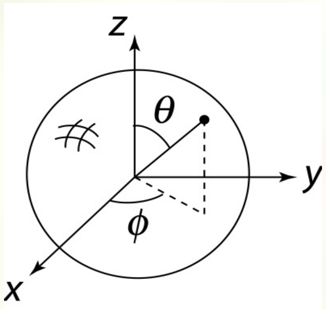

- Implicit $g(x,y,z) = 0$

- Parametric $S(u,v) = (x(u,v), y(u,v), z(u,v))$
  - For the sphere:

$$x(u,v) = r \cos 2\pi v \sin \pi u$$

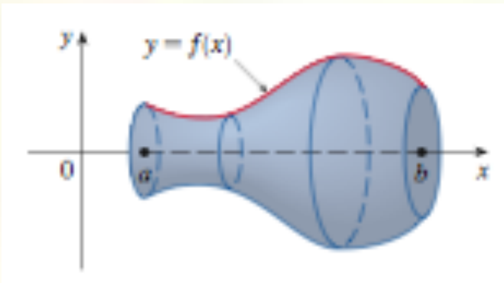
$$y(u,v) = r \sin 2\pi v \sin \pi u$$

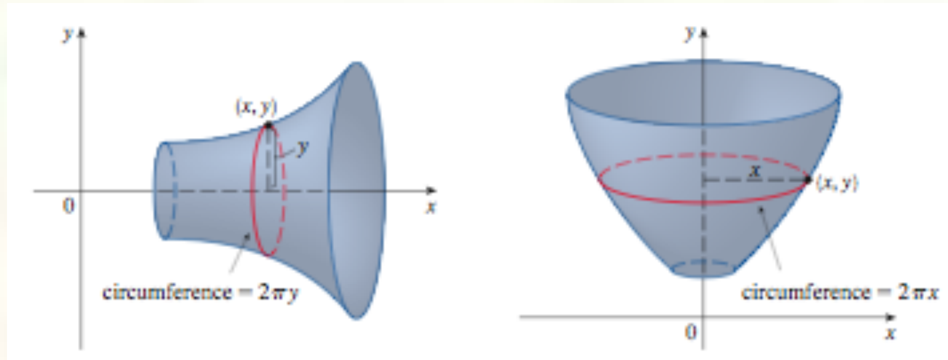
$$z(u,v) = r \cos \pi u$$




As with curves, we'll focus on parametric surfaces.

# Surfaces of revolution




- Idea: rotate a 2D **profile curve** around an axis.
- What kinds of shapes can you model this way?
- **Find:** A surface $S(u,v)$ which is radius($z$) rotated about the $z$ axis.
- **Solution:**

$$x = \text{radius}(u)\cos(v)$$

$$y = \text{radius}(u)\sin(v)$$

$$z = u \qquad\qquad u \in [z_{\min}, z_{\max}], \quad v \in [0, 2\pi]$$

# Extruded surfaces

- **Given:** A curve $C(u)$ in the $xy$-plane:

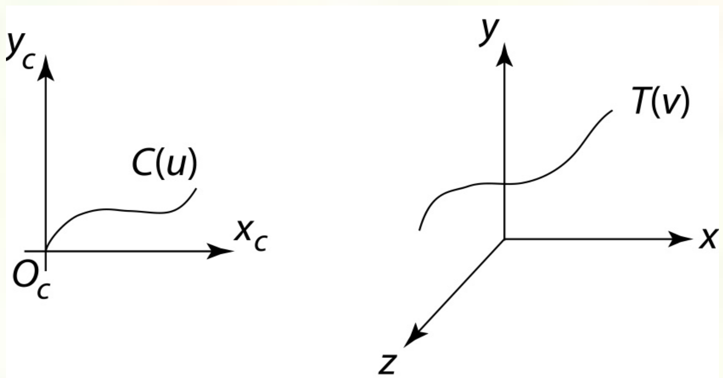
$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

- **Find:** A surface $S(u,v)$ which is $C(u)$ extruded along the $z$ axis.
- **Solution:**

$$\begin{aligned} x &= c_x(u) \\ y &= c_y(u) \qquad u \in [u_{\min}, u_{\max}], \quad v \in [z_{\min}, z_{\max}] \\ z &= v \end{aligned}$$

# General sweep surfaces

- The **surface of revolution** is a special case of a **swept surface**.
- Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



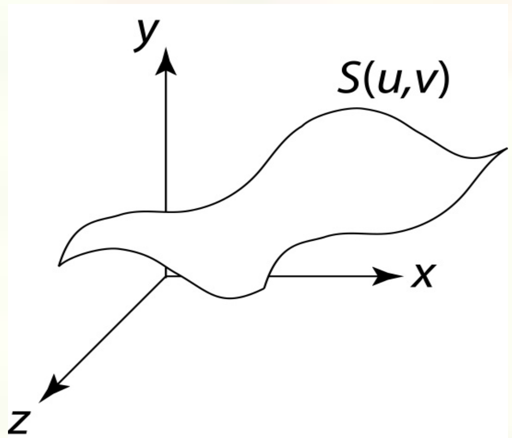

- More specifically:
  - Suppose that $C(u)$ lies in an $(x_c,y_c)$ coordinate system with origin $O_c$.
  - For every point along $T(v)$, lay $C(u)$ so that $O_c$ coincides with $T(v)$.

# Orientation

- The big issue:
  - How to orient $C(u)$ as it moves along $T(v)$?

- Here are two options:

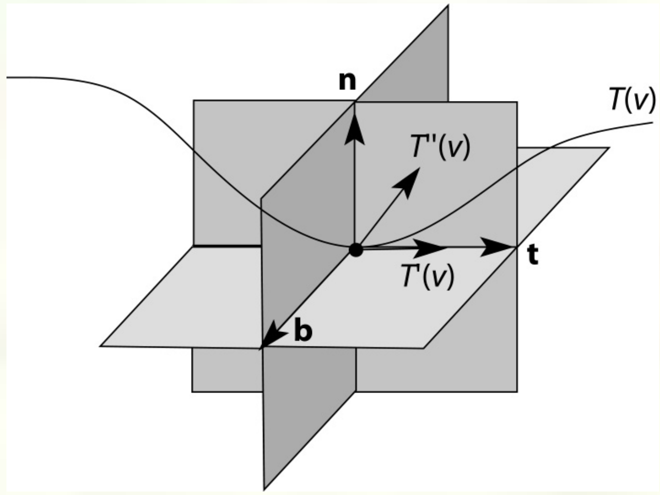
1. **Fixed** (or **static**): Just translate $O_c$ along $T(v)$.




2. Moving. Use the **Frenet frame** of $T(v)$.
   - Allows smoothly varying orientation.
   - Permits surfaces of revolution, for example.

# Frenet frames

- Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.




- To get a 3D coordinate system, we need 3 independent direction vectors.

$$\mathbf{t}(v) = \text{normalize}[T'(v)]$$

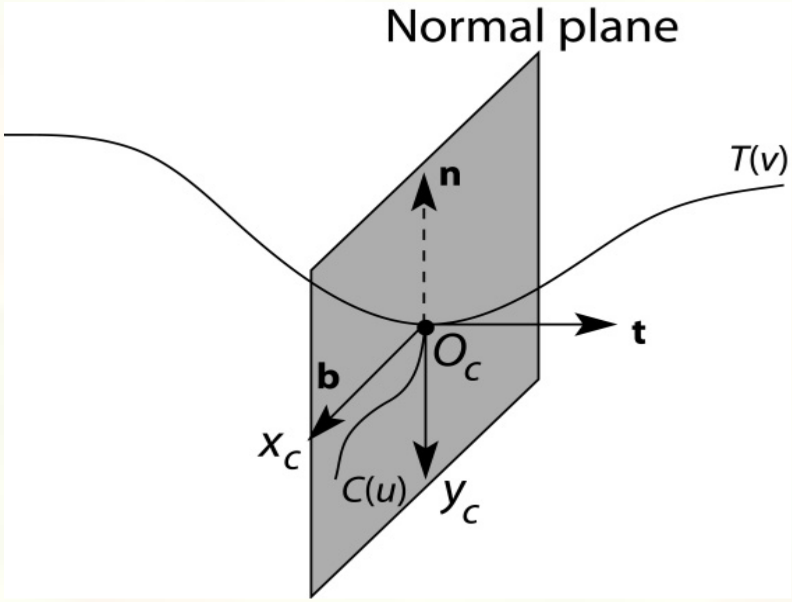
$$\mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$$

$$\mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$$

- As we move along $T(v)$, the Frenet frame $(\mathbf{t},\mathbf{b},\mathbf{n})$ varies smoothly.

# Frenet swept surfaces

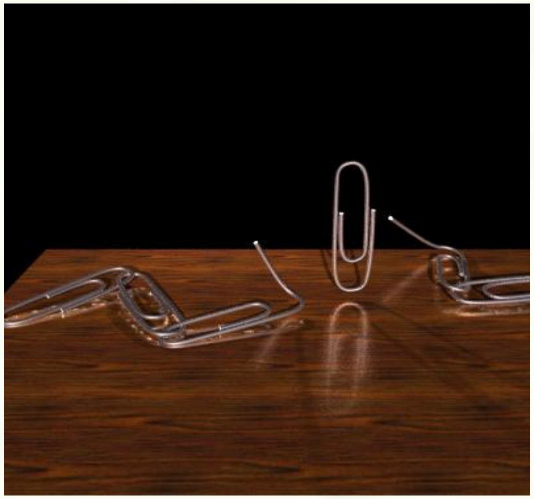
- Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:
  - Put $C(u)$ in the **normal plane** .
  - Place $O_c$ on $T(v)$.
  - Align $x_c$ for $C(u)$ with **b**.
  - Align $y_c$ for $C(u)$ with **-n**.

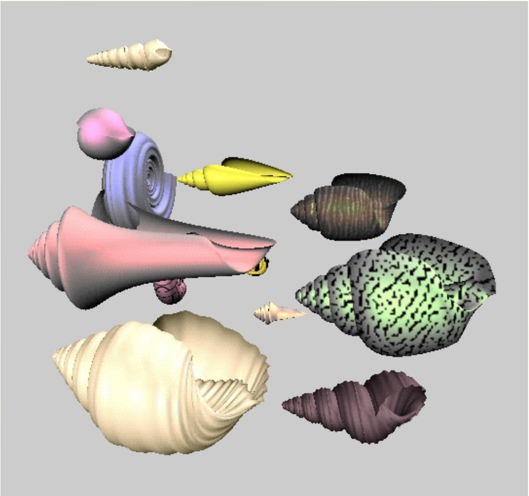



- If $T(v)$ is a circle, you get a surface of revolution exactly!
- What happens at inflection points, i.e., where curvature goes to zero?

# Variations

- Several variations are possible:
  - Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
  - Morph $C(u)$ into some other curve $\overline{C}(u)$ as it moves along $T(v)$.
  - …





# Generalizing from Parametric Curves

- Flashback to curves:

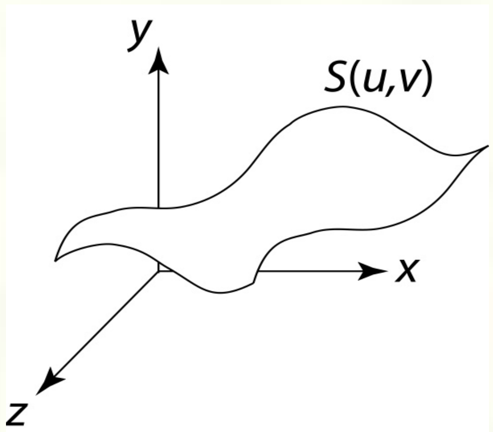
  We directly defined parametric function
  f($u$), as a cubic polynomial.

- Why a cubic polynomial?

  - minimum degree for C2 continuity
  - "well behaved"

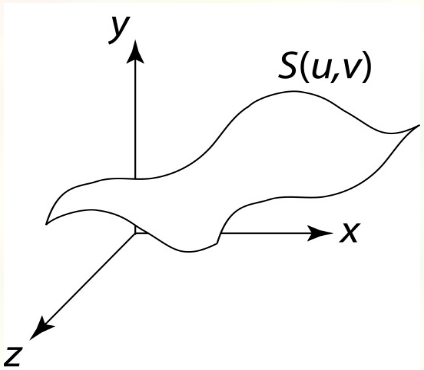
- Can we do something similar for surfaces?

Initially, just think of a height field: height = f($u$,$v$).



# Cubic patches

Cubics curves are good… Let's extend them in the obvious way to surfaces:




$$f(u) = 1 + u + u^2 + u^3$$

$$g(v) = 1 + v + v^2 + v^3$$

$$f(u)g(v) = 1 + u + v + uv + u^2 + v^2 + uv^2 + vu^2 + \ldots + u^3v^3$$

16 terms in this function.

Let's allow the user to pick the coefficient for each of them:

$$f(u)g(v) = c_0 + c_1u + c_2v + \ldots + c_{15}u^3v^3$$

# Interesting properties

$$f(u,v) = c_0 + c_1 u + c_2 v + \ldots + c_{15} u^3 v^3$$

What happens if I pick a particular '$u$'?

$$f(u,v) =$$

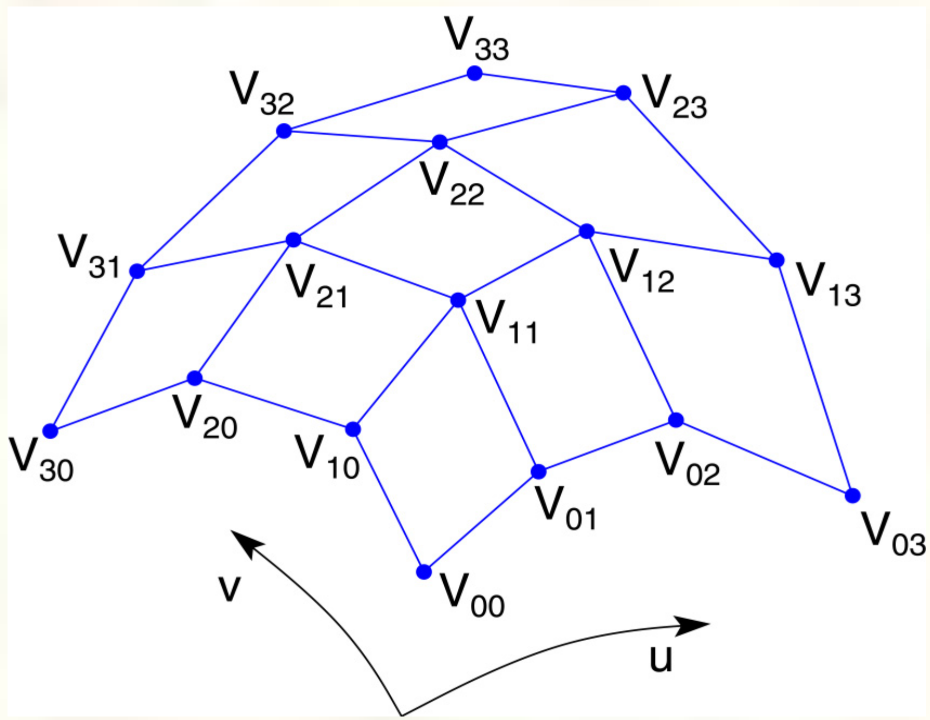
What happens if I pick a particular '$v$'?

$$f(u,v) =$$

What do these look like graphically on a patch?

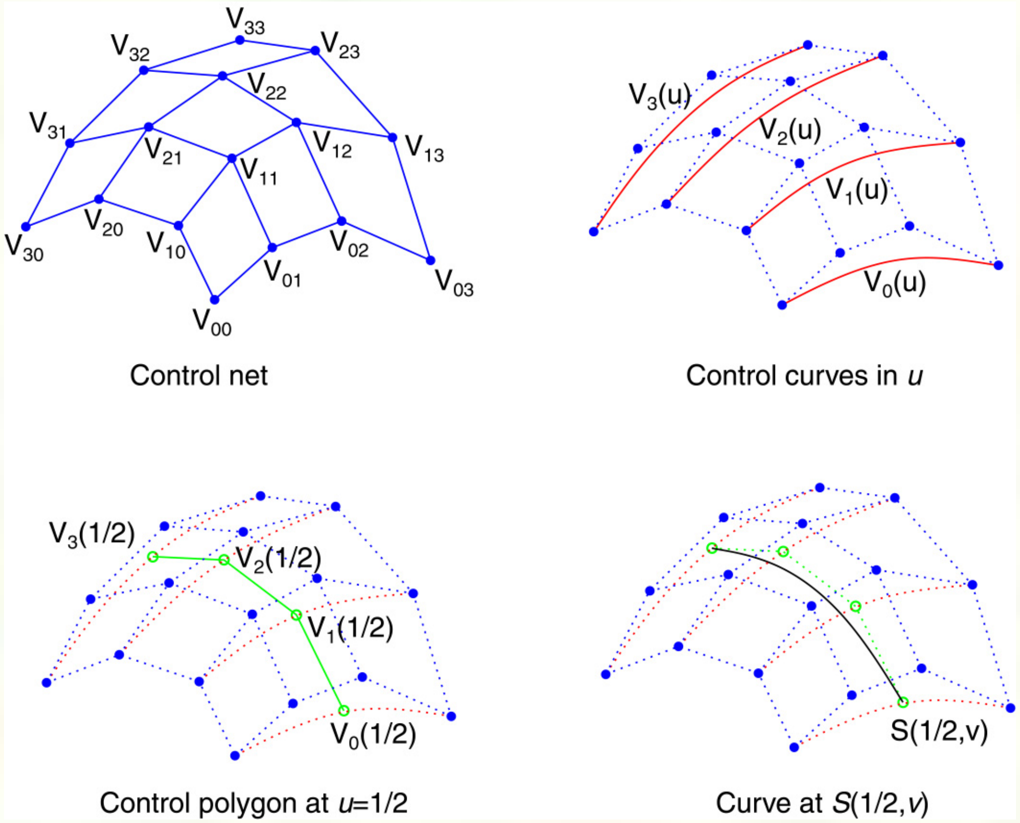
# Use control points

- As before, directly manipulating coefficients is not intuitive.
  - Instead, directly manipulate control points.
  - These control points indirectly set the coefficients, using approaches like those we used for curves.



# Tensor product Bézier surface

- Let's walk through the steps:



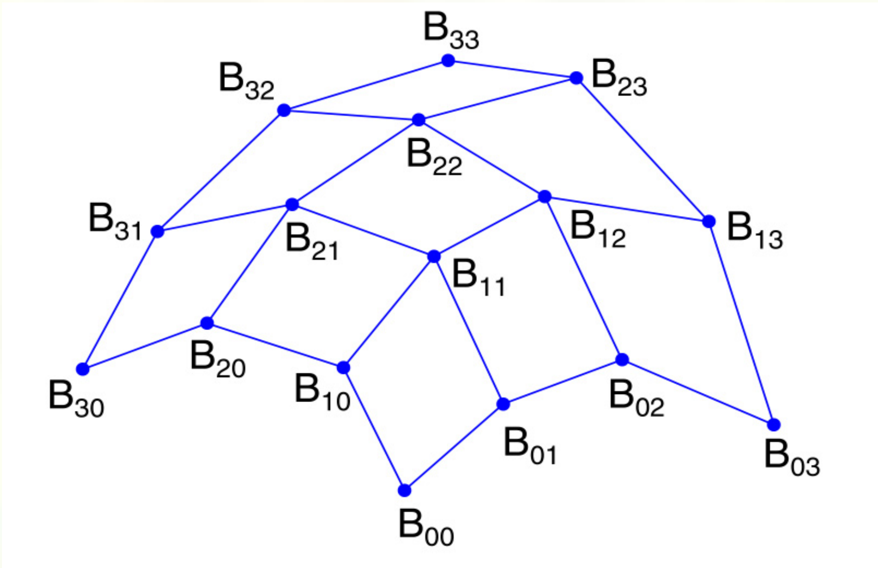

- Which control points are interpolated by the surface?

# Matrix form of Bézier surfaces

- Recall that Bézier curves can be written in terms of the Bernstein polynomials:
$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathrm{B}_{i,n}(u)\,\mathbf{p}_i$$
- They can also be written in a matrix form:
$$\mathbf{p}(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{U}\mathbf{M}_{\mathrm{B}}\mathbf{P}$$
- Tensor product surfaces can be written out similarly:
$$\mathbf{p}(u) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathrm{B}_{i,n}(u)\mathrm{B}_{j,n}(v)\,\mathbf{p}_{i,j}$$
$$= \mathbf{U}\mathbf{M}_{\mathrm{B}}\mathbf{P}_{\mathrm{s}}\mathbf{M}_{\mathrm{B}}^{T}\mathbf{V}^{T}$$

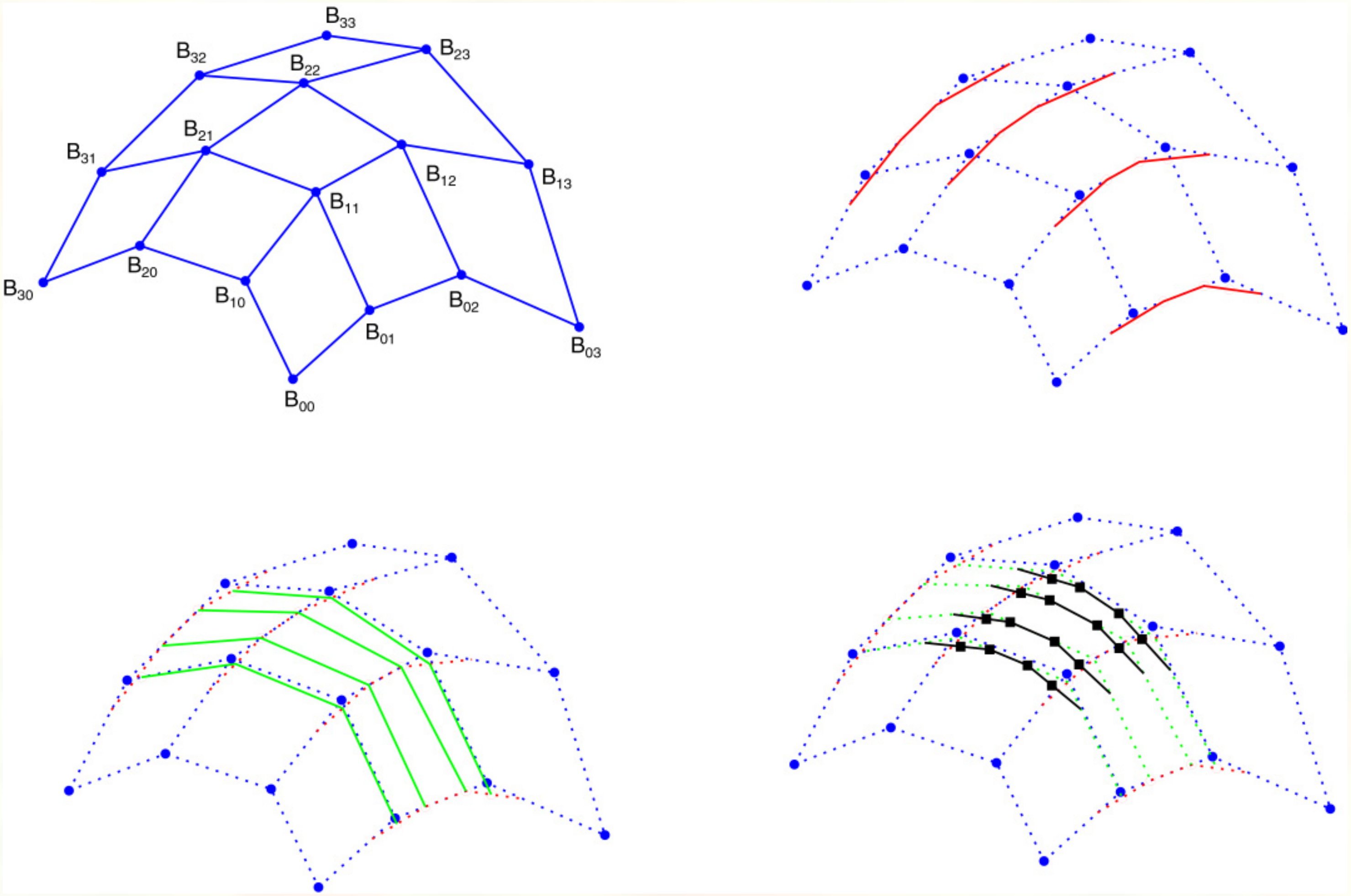
# Tensor product B-spline surfaces

- As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce $C^2$ continuity and local control, we get B-spline curves:




  - treat rows of $B$ as control points to generate Bézier control points in $u$.
  - treat Bézier control points in $u$ as B-spline control points in $v$.
  - treat B-spline control points in $v$ to generate Bézier control points in $u$.

# Tensor product B-spline surfaces

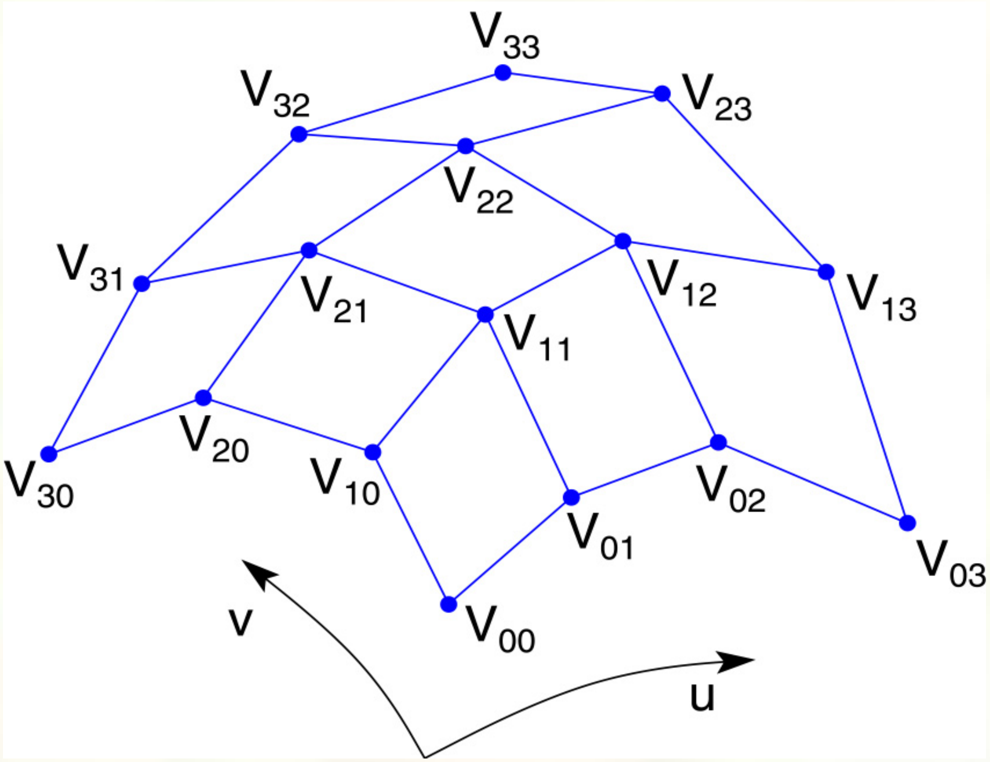



Which B-spline control points are interpolated by the surface?

# Continuity for surfaces

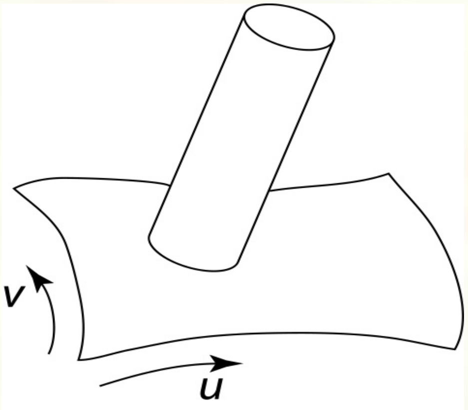
Continuity is more complex for surfaces than curves. Must examine partial derivatives at patch boundaries.

$G^1$ continuity refers to tangent plane.



# Trimmed NURBS surfaces

- Uniform B-spline surfaces are a special case of NURBS surfaces.
- Sometimes, we want to have control over which parts of a NURBS surface get drawn.
- For example:




- We can do this by **trimming** the *u*-*v* domain.
  - Define a closed curve in the *u*-*v* domain (a **trim curve**)
  - Do not draw the surface points inside of this curve.
- It's really hard to maintain continuity in these regions, especially while animating.

# Next class: Subdivision surfaces

- **Topic:**

  How do we extend ideas from subdivision curves to the problem of representing surfaces?

- **Recommended Reading:**

  - Stollnitz, DeRose, and Salesin. Wavelets for Computer Graphics: Theory and Applications, 1996, section 10.2.
    [Course reader pp. 262-268]