CS384G **Homework 1** **Fall 2006**

Name: ______________________________

Sampling theory, image processing, affine transformations
Assigned: Sept 19, 2006
Due: Oct 3, 2006
(at the beginning of class)

Directions: Please provide short written answers to the following questions. Feel free to talk over the problems in general terms with classmates, but please answer the questions on your own.

1. (40 points) **Fourier transforms and signal reconstruction**

   In this problem, you will take a closer look at convolution, Fourier transforms and signal reconstruction. We begin with a couple of preliminaries.

   i) Recall that the box function $\text{II}(x)$ is defined as:

   $$\text{II}(x) = \begin{cases} 1 & |x| < 1/2, \\ 1/2 & |x| = 1/2, \\ 0 & |x| > 1/2. \end{cases}$$

   and its Fourier transform is $\text{sinc}(s) = \sin(\pi s)/\pi s$. Likewise, the Fourier transform of $\text{sinc}(x)$ is $\text{II}(s)$. More generally, for some constant frequency ω_0, the Fourier transform of $\text{II}(\omega_0 x)$ is $\frac{1}{\omega_0}\text{sinc}(\frac{1}{\omega_0}s)$ Also, we can show that $\text{sinc}(0) = 1$. This transform pair is depicted graphically in the lecture notes.

   ii) The dirac delta function $\delta(x)$ can be defined as:

   $$\begin{aligned} \delta(x) &= 0, \quad \text{for all } x \neq 0, \\ \delta(0) &= \infty, \end{aligned}$$

   and with the property that $\int_{-\infty}^{\infty} \delta(x)dx = 1$.
   The Fourier transform of $\cos(2\pi f_0 x)$ is

   $$\frac{1}{2}\delta(x - f_0) + \frac{1}{2}\delta(x + f_0),$$

   and vice-versa. The constant f_0 represents the frequency of the cosine function. This transform pair is also depicted graphically in the lecture notes.

   iii) If the Fourier transform of $h(x)$ is $H(s)$, then the Fourier transform of $kh(x)$ is $kH(s)$.

   iv) The hat function, $\wedge(x)$, is defined as:

   $$\wedge(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & |x| > 1. \end{cases}$$

We sample a function by multiplying with the comb or shah function:

$$\hat{f}(x) = f(x)\text{III}(x) = \sum_{i=-\infty}^{i=\infty} f(i)\delta(x - i)$$

We reconstruct by convolving with a reconstruction filter $r(x)$:

$$\tilde{f}(x) = r(x) * \hat{f}(x) = \sum_{i=-\infty}^{i=\infty} \hat{f}(i)r(x - i)$$

Now we get to the problems you need to solve. (Hint: There is an easy way and a hard way to do these problems. The easy way is to use various simple properties of the Fourier transform.)

a) Convolution:
Given a function $f(x) = \cos(2\pi x) + \cos(8\pi x)$

i) What is its Fourier transform (in equation form)? Plot the magnitude of the Fourier transform, with properly labeled axes.

ii) Given a convolution filter $h(x) = \text{sinc}(5x)$, what is the function that results from convolving $f(x)$ with $h(x)$?

b) Sampling and Reconstruction:
Given the same function $f(x) = \cos(2\pi x) + \cos(8\pi x)$

i) Plot this function.

ii) If we sample this signal at points 0, 1/8, 2/8, 3/8, ., and then reconstruct it with a linear reconstruction kernel, $h(x) = \wedge(8x)$, what is the plot of the reconstructed signal? Is this signal the same as the original signal $f(x)$? Discuss why or why not.

2. (?? points) **Pre-filtering to minimize aliasing during sampling**
(note: This is the hardest problem on the assignment)

This problem considers the use of pre-filtering to reduce the high-frequency content of a signal prior to sampling it. By pre-filtering, we can reduce or eliminate the aliasing introduced by sampling.

To keep the problem simple, we are considering 1D sampling of audio (temporal) signals rather than 2D sampling of image (spatial) signals. Thus, in the Fourier domain the units are Hz (temporal cycles/second). But these same ideas are also used for 2D and 3D sampling in computer graphics.

Digital telephone systems sample audio signals at a rate of 8kHz.

a) What is the highest audio frequency that can be sampled by a telephone system without aliasing?

b) Humans can hear frequencies up to approximately 20 kHz. For the sake of simplicity, assume that the audio signal that we are sampling has a flat frequency spectrum from 0 to 20KHz. To avoid aliasing when we sample this signal, we are going to first pre-filter it with an analog filter to reduce the amount of energy in the signal above the Nyquist frequency. Our filter operates in the time domain (equivalent to the spatial domain for images). Our filter is implemented as convolution with a Gaussian in the time domain. Express the filter mathematically in both the time domain and the frequency domain. i.e. what is $h(t)$ and what is $H(\omega)$? The width of $h(t)$ should be expressed in terms of a standard deviation σ.

c) Because the Gaussian never becomes exactly zero, we cannot use a Gaussian as a perfect pre-filter. But, if we choose the width of the Gaussian well it can be a good filter. Suppose that we want to insure that 98% of the energy in the sampled signal is not aliased (i.e. 98% of the energy in the sampled signal comes from frequencies that were below the Nyquist frequency in the original audio signal). What width of Gaussian should we use? (Hint, you are going to have to consider area under the curve $H(\omega)$.)

d) With the filter from part (c), what fraction of the original signal energy *below* the Nyquist frequency are we discarding? Ideally the answer would be 0%, but the answer is higher than that because the Gaussian is not a perfect low-pass filter.

3. (25 points) **Image processing**

   Describe the effect of each of the following filters. In addition, indicate which filter will cause the most blurring and which, when convolved with a solid (positive) intensity image, will produce the brightest image and which will produce the darkest image. Justify your answers.

0.1	0.1	0.1
0.1	0.1	0.1
0.1	0.1	0.1

0	0	1
0	-2	0
1	0	0

0	0.2	0
0.2	0.4	0.2
0	0.2	0

0	-1	0
0	3	0
0	-1	0

4. (35 points) **Affine Transformations**

a) As discussed in class, any three-dimensional affine transformation can be represented with a 4×4 matrix. Match each of the matrices below to exactly one of the following transformations (not all blanks will be filled):

____ Differential (Non-Uniform) Scaling
____ Reflection
____ Rotation about the z-axis with non-uniform scaling
____ Rotation about the y-axis with non-uniform scaling
____ Translation
____ Rotation about the x-axis
____ Rotation about the y-axis
____ Rotation about the z-axis
____ Shearing along z with respect to the x-y plane (z=0 plane unchanged by shear)
____ Shearing along x with respect to the y-z plane (x=0 plane unchanged by shear)
____ Rotation about the x-axis and translation
____ Uniform scaling
____ Reflection with uniform scaling

$$A = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0.6 & 0.8 & 0 \\ 0 & -0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Consider a line that passes through a point $\mathbf{p} = (p_x, p_y, p_z)$ in the direction $v = (\cos(\alpha), 0, \sin(\alpha))$. Write out the product of matrices that would perform a rotation by θ about this line. You should not multiply these matrices out, but you do need to write out all of the elements in these matrices.