14. Subdivision curves

Reading

Recommended:

- Stollnitz, DeRose, and Salesin. *Wavelets for Computer Graphics: Theory and Applications,* 1996, section 6.1-6.3, A.5.

Note: there is an error in Stollnitz, et al., section A.5. Equation A.3 should read:

$$\mathbf{MV} = \mathbf{V}\Lambda$$

Subdivision curves

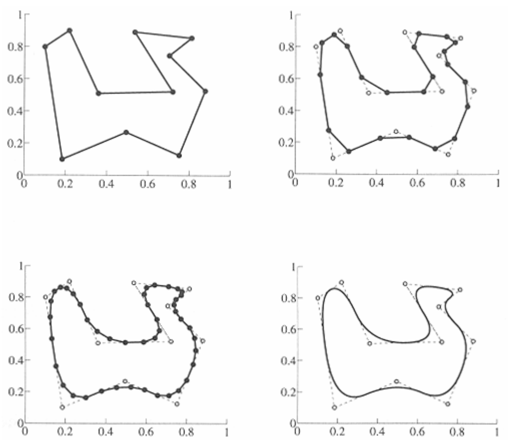
Idea:

- repeatedly refine the control polygon

$$P^1 \rightarrow P^2 \rightarrow P^3 \rightarrow \cdots$$

- curve is the limit of an infinite process

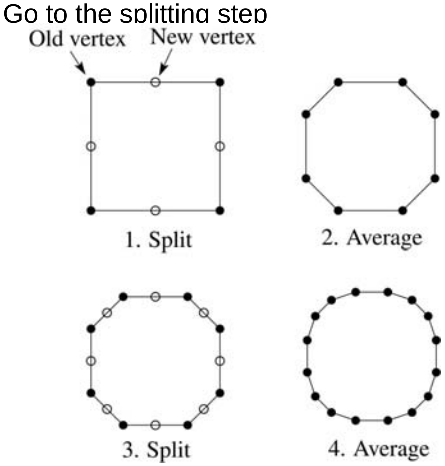
$$Q = \lim_{j \rightarrow \infty} P^j$$



Chaikin's algorithm

Chakin introduced the following "corner-cutting" scheme in 1974:

- Start with a piecewise linear curve
- Insert new vertices at the midpoints (the **splitting step**)
- Average each vertex with the "next" (clockwise) neighbor (the **averaging step**)
- Go to the splitting step



Averaging masks

The limit curve is a quadratic B-spline!

Instead of averaging with the nearest neighbor, we can generalize by applying an **averaging mask** during the averaging step:

$$r = (\ldots, r_{-1}, r_0, r_1, \ldots)$$

In the case of Chaikin's algorithm:

$$r =$$

5

Can we generate other B-splines?

Answer: Yes

Lane-Riesenfeld algorithm (1980)

Use averaging masks from Pascal's triangle:

$$r = \frac{1}{2^n}\left(\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}\right)$$

Gives B-splines of degree n+1.

n=0: **1**

n=1:

1

1 1

n=2:

1

1 1

1 2 1

6

Subdivide ad nauseum?

After each split-average step, we are closer to the **limit curve**.

How many steps until we reach the final (limit) position?

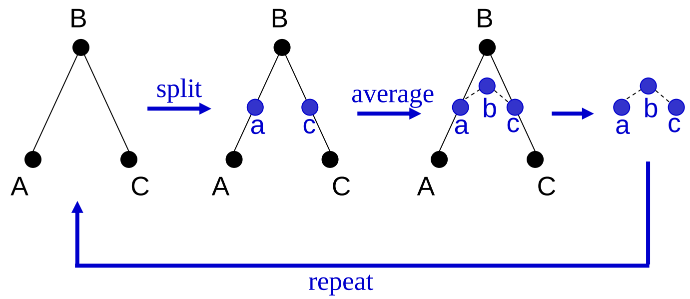
Can we push a vertex to its limit position without infinite subdivision? Yes!

One subdivision step

Consider the cubic B-spline subdivision mask:

$$\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$$

Now consider what happens during splitting and averaging:

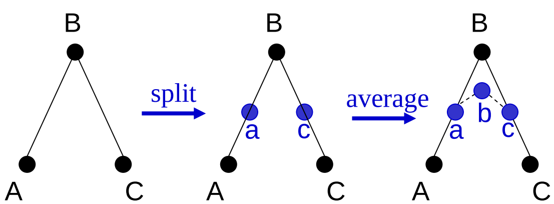


Math for one subdivision step

Subdivision mask:

$$\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$$

One subdivision step:



Split: $\mathbf{a} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) \qquad \mathbf{c} = \frac{1}{2}(\mathbf{B}+\mathbf{C})$

Average:

a and **c** do not change

$$\mathbf{b} = \frac{1}{4}(\mathbf{a}+2B+\mathbf{c}) \qquad = \frac{1}{8}(\mathbf{A}+6\mathbf{B}+\mathbf{C})$$

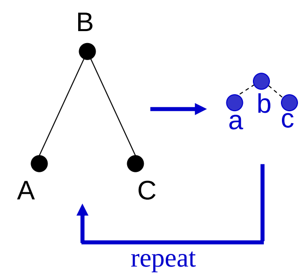
9

Consolidated math for one step

Subdivision mask:

$$\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$$

One subdivision step:



Consolidated math for one subdivision step:

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{bmatrix} = \frac{1}{8}\begin{bmatrix}4 & 4 & 0\\ 1 & 6 & 1\\ 0 & 4 & 4\end{bmatrix}\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{bmatrix}$$

P_{j+1} Local subdivision matrix 'S' P_j

10

Local subdivision matrix, cont'd

Tracking just the *x* components through subdivision:

$$P_j = SP_{j-1} = S \cdot SP_{j-2} = S \cdot S \cdot SP_{j-3} = \cdots = S^j P_0$$

The limit position of the x's is then:

$$P_\infty = S^\infty P_0$$

or as we'd say in calculus…

$$P_\infty = \lim_{j \to \infty} S^j P_0$$

OK, so how do we apply a matrix an infinite number of times??

Eigenvectors and eigenvalues

To solve this problem, we need to look at the eigenvectors and eigenvalues of *S*. First, a review…

Let *v* be a vector such that:

$$Sv = \lambda v$$

We say that *v* is an eigenvector with eigenvalue λ.

An *n*x*n* matrix can have *n* eigenvalues and eigenvectors:

$$Sv_1 = \lambda_1 v_1$$
$$\vdots$$
$$Sv_n = \lambda_n v_n$$

If the eigenvectors are linearly independent (which means that *S* is *non-defective*), then they form a basis, and we can re-write *P* in terms of the eigenvectors:

$$P = \sum_i^n a_i v_i$$

To infinity, but not beyond…

Now let's apply the matrix to the vector X:

$$P_1 = SP_0 = S\sum_i^n a_i v_i = \sum_i^n a_i S v_i = \sum_i^n a_i \lambda_i v_i$$

Applying it *j* times:

$$P_j = S^j P_0 = S^j \sum_i^n a_i v_i = \sum_i^n a_i S^j v_i = \sum_i^n a_i \lambda_i^j v_i$$

Let's assume the eigenvalues are non-negative and sorted so that:

$$\lambda_1 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

Now let *j* go to infinity:

$$P_\infty = \lim_{j\to\infty} S^j P_0 = \lim_{j\to\infty} \sum_i^n a_i \lambda_i^j v_i$$

If $\lambda_1 > 1$, then:

If $\lambda_1 < 1$, then:

If $\lambda_1 = 1$, then:

Evaluation masks

What are the eigenvalues and eigenvectors of our cubic B-spline subdivision matrix?

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2} \qquad \lambda_3 = \frac{1}{4}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

We're OK!

But what is the final position?

$$P_\infty = \lim_{j\to\infty} \left(a_1 \lambda_1^j v_1 + a_2 \lambda_2^j v_2 + a_3 \lambda_3^j v_3 \right)$$

$$P_\infty =$$

Almost done… from earlier we know that we can find 'a', we but didn't give specifics.

Evaluation masks, cont'd

To finish up, we need to compute a_1.

Remember: $P_0 = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

Rewrite as:

$$P_0 = \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{V}A$$

We need to solve for the vector 'A'.
(This is really just a change of basis for representing the vector P). The solution is:

$$A = \mathbf{V}^{-1} P_0$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \cdots & u_1^T & \cdots \\ \cdots & u_2^T & \cdots \\ & \vdots & \\ \cdots & u_n^T & \cdots \end{bmatrix} P_0$$

Now we can compute the limit position:

$$P_\infty = a_1 = u_1^T P_0$$

We call u_1 the **evaluation mask**.

Evaluation masks, cont'd

Note that we need not start with the 0^{th} level control points and push them to the limit.

If we subdivide and average the control polygon j times, we can push the vertices of the refined polygon to the limit as well:

$$P_\infty = S^\infty P_j = u_1^T P_j$$

So far we've been looking at math for a subdivision function f(x).

For a 2D parametric subdivision curve, (x(u), y(u)), just apply these formulas separately for the x(u) and y(u) functions.

Recipe for subdivision curves

The evaluation mask for the cubic B-spline is:

$$\frac{1}{6}\begin{pmatrix}1 & 4 & 1\end{pmatrix}$$

Now we can cook up a simple procedure for creating subdivision curves:

- Subdivide (split+average) the control polygon a few times. Use the averaging mask.
- Push the resulting points to the limit positions. Use the evaluation mask.

17

Derivative of subdiv. function

What is the tangent to the cubic B-spline function?

Consider the formula for P again:

$$P_j = a_1\lambda_1^j v_1 + a_2\lambda_2^j v_2 + a_3\lambda_3^j v_3$$

$$P_j = a_1(1)^j\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2(\frac{1}{2})^j\begin{pmatrix}-1\\0\\1\end{pmatrix} + a_3(\frac{1}{4})^j\begin{pmatrix}2\\-1\\2\end{pmatrix}$$

Where:

$$P_j = \begin{bmatrix}left\\center\\right\end{bmatrix}$$

Derivative is just:

$$P' = \lim_{j\to\infty}\frac{center-left}{\Delta x} = \lim_{j\to\infty}\frac{center-left}{\frac{1}{2^j}}$$

$$P' = \lim_{j\to\infty}\left(a_2\left(\frac{1}{2}\right)^j\frac{0+1}{\frac{1}{2^j}}\right) = a_2 = u_2^T P_0$$

18

Tangent analysis for 2D curve

What is the tangent to a parametric cubic B-spline **2D curve**?

Using a similar derivation to what we just did for a 1D function (but omitting details):

$$\mathbf{t} = \lim_{j\to\infty} \frac{P_{Center,j} - P_{Left,j}}{\left\| P_{Center,j} - P_{Left,j} \right\|}$$

$$= \frac{u_2^T P_0}{\left\| u_2^T P_0 \right\|}$$

Thus, we can compute the tangent using the *second* left eigenvector! This analysis holds for general subdivision curves and gives us the **tangent mask**.

Approximation vs. Interpolation of Control Points

Previous subdivision scheme *approximated* control points. Can we *interpolate* them?

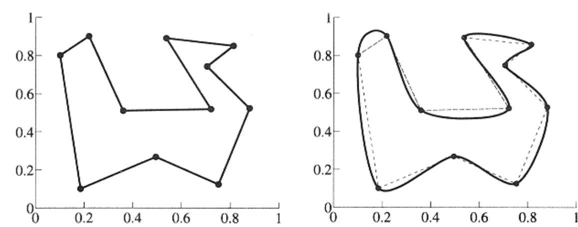
Yes: **DLG interpolating scheme (1987)**

Slight modification to subdivision algorithm:

- splitting step introduces midpoints
- averaging step *only changes midpoints*

For DLG (Dyn-Levin-Gregory), use:

$$r = \frac{1}{16}(-2, 5, 10, 5, -2)$$



Since we are only changing the midpoints, the points after the averaging step do not move.

Next time: Animation Principles

Topic:

How does an artist make
a "good" animation?

Read:

- John Lasseter. Principles of traditional animation applied to 3D computer animation. SIGGRAPH 1987. [Course reader pp. 295-304]

Recommended:

- Frank Thomas and Ollie Johnston, Disney animation: The Illusion of Life, Hyperion, 1981.