Fourier Transforms

Fourier series

- To go from f(θ) to f(t) substitute $\theta = \frac{2\pi}{T} t = \omega_0 t$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- To deal with the first basis vector being of length 2π instead of π, rewrite as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Fourier series

- The coefficients become

$$a_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin(k\omega_0 t)\, dt$$

Fourier series

- Alternate forms

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (\cos(n\omega_0 t) + \frac{b_n}{a_n} \sin(n\omega_0 t))$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (\cos(n\omega_0 t) - \tan(\varphi_n) \sin(n\omega_0 t))$$

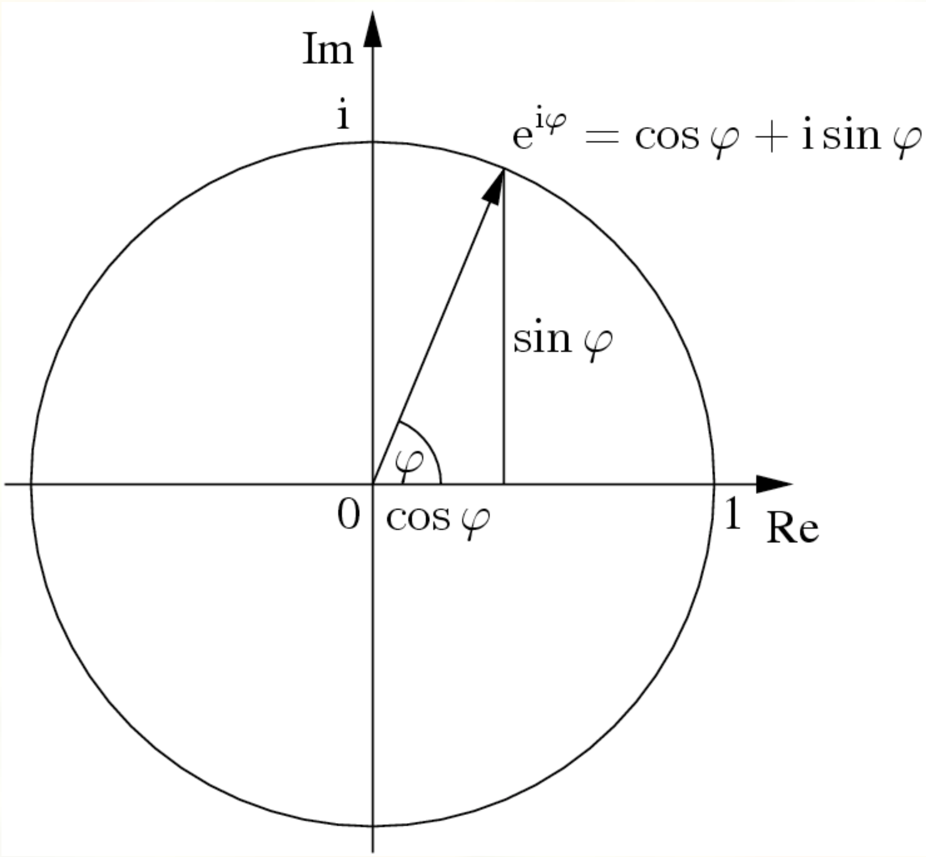
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \varphi_n)$$

- where

$$c_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \varphi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Complex exponential notation

- Euler's formula $e^{ix} = \cos(x) + i\sin(x)$



Phasor notation:

$$x + iy = |z|e^{i\varphi}$$

where $|z| = \sqrt{x^2 + y^2}$

$$= \sqrt{z\bar{z}}$$

$$= \sqrt{(x+iy)(x-iy)}$$

and $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

Euler's formula

- Taylor series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

- Even function (f(x) = f(-x))

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

- Odd function (f(x) = -f(-x))

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots$$

$$= \cos(x) + i\sin(x)$$

Complex exponential form

- Consider the expression

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} F_n \cos(n\omega_0 t) + iF_n \sin(n\omega_0 t)$$

$$= \sum_{n=0}^{\infty} (F_n + F_{-n}) \cos(n\omega_0 t) + i(F_n - F_{-n}) \sin(n\omega_0 t)$$

- So $a_n = F_n + F_{-n}$ and $b_n = i(F_n - F_{-n})$
- Since a_n and b_n are real, we can let $F_{-n} = \overline{F_n}$ and get $a_n = 2\,\mathrm{Re}(F_n)$ and $b_n = -2\,\mathrm{Im}(F_n)$

$$\mathrm{Re}(F_n) = \frac{a_n}{2} \quad \text{and} \quad \mathrm{Im}(F_n) = -\frac{b_n}{2}$$

Complex exponential form

- Thus

$$F_n = \frac{1}{T}\left(\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt - i\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt \right)$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)(\cos(n\omega_0 t)\,dt - i\sin(n\omega_0 t))dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,e^{-in\omega_0 t}\,dt$$

$$= \left|F_n\right| e^{i\varphi_n}$$

- So you could also write $f(t) = \sum_{n=-\infty}^{\infty} \left|F_n\right| e^{i(n\omega_0 t + \varphi_n)}$

Fourier transform

- We now have $$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, e^{-in\omega_0 t}\, dt$$

- Let's not use just discrete frequencies, $n\omega_0$, we'll allow them to vary continuously too

- We'll get there by setting $t_0=-T/2$ and taking limits as T and n approach ∞

Fourier transform

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} e^{in\omega_0 t} \frac{1}{T} \int_{T/2}^{T/2} f(t)\, e^{-in\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{T}t} \frac{2\pi}{T} \frac{1}{2\pi} \int_{T/2}^{T/2} f(t)\, e^{-in\frac{2\pi}{T}t} dt$$

$$\lim_{T\to\infty}\left(\frac{2\pi}{T}\right) = d\omega \qquad \lim_{n\to\infty} n\, d\omega = \omega$$

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \right] d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega$$

Fourier transform

- So we have (unitary form, angular frequency)

$$\mathcal{F}(f(t)) = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$\mathcal{F}^{-1}(F(\omega)) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

- Alternatives (Laplace form, angular frequency)

$$\mathcal{F}(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$\mathcal{F}^{-1}(F(\omega)) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

Fourier transform

- Ordinary frequency

$$s = \frac{\omega}{2\pi}$$

$$\mathsf{F}(f(t)) = F(s) = \int_{-\infty}^{\infty} f(t) e^{-i 2\pi s t} dt$$

$$\mathsf{F}^{-1}(F(s)) = f(t) = \int_{-\infty}^{\infty} F(\phi) e^{i 2\pi s t} ds$$

Fourier transform

- Some sufficient conditions for application
  - Dirichlet conditions
    - $\int_{-\infty}^{\infty} |f(t)| dt < \infty$
    - *f(t)* has finite maxima and minima within any finite interval
    - *f(t)* has finite number of discontinuities within any finite interval
  - Square integrable functions (L^2 space)

    $$\int_{-\infty}^{\infty} [f(t)]^2 dt < \infty$$
  - Tempered distributions, like Dirac delta

    $$\mathsf{F}(\delta(t)) = \frac{1}{\sqrt{2\pi}}$$

Fourier transform

- Complex form – orthonormal basis functions for space of tempered distributions

$$\int_{-\infty}^{\infty} \frac{e^{i\omega_1 t}}{\sqrt{2\pi}} \frac{e^{-i\omega_2 t}}{\sqrt{2\pi}} dt = \delta(\omega_1 - \omega_2)$$