Splines

Spline Curves

- Successive linear blend
- Basis polynomials
- Recursive evaluation
- Properties
- Joining segments

Tensor-product-patch Spline Surfaces

- Tensor product patches
- Evaluation
- Properties
- Joining patches

Triangular-patch Spline Surfaces

- Coordinate frames and barycentric frames
- Triangular patches

Discontinuities

- Basis polynomials
- Multiple segments
- Basis splines

Continuities

- Combining basis splines for smoothness
- Curves with basis splines

B-Splines

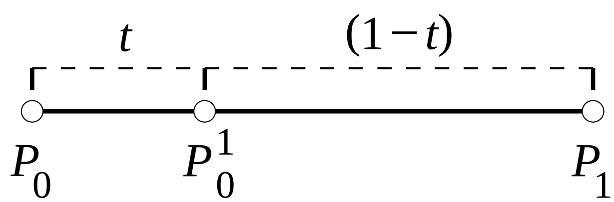
- General segmentation and smoothness
- Knots and evaluation

Constructing Curve Segments

Linear blend:

- Line segment from an affine combination of points

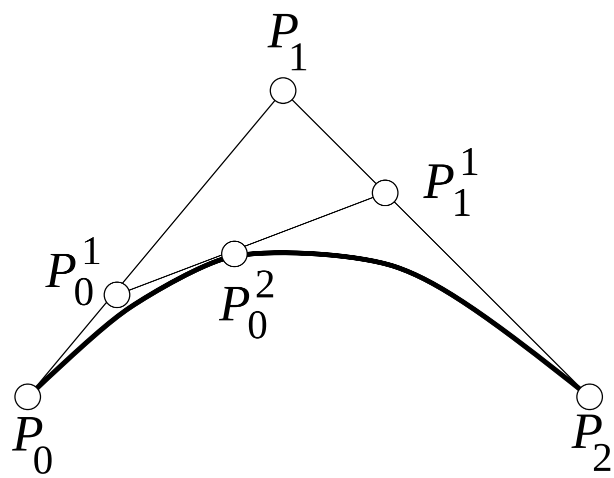
$$P_0^1(t) = (1-t)P_0 + tP_1$$



Quadratic blend:

- Quadratic segment from an affine combination of line segments

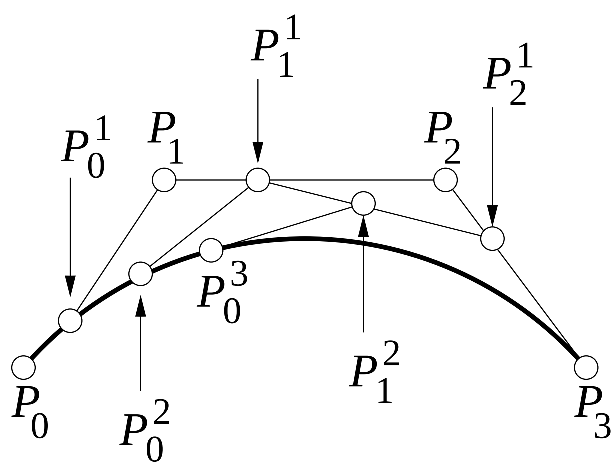
$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t)
\end{aligned}
$$



Cubic blend:

- Cubic segment from an affine combination of quadratic segments

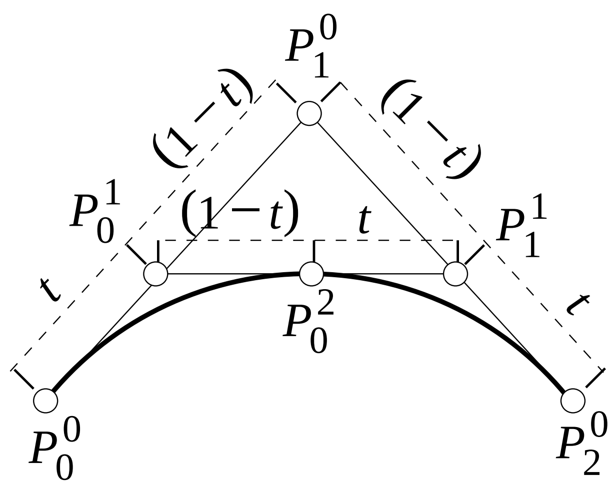
$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t) \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_2^1(t) &= (1-t)P_2 + tP_3 \\
P_1^2(t) &= (1-t)P_1^1(t) + tP_2^1(t) \\
P_0^3(t) &= (1-t)P_0^2(t) + tP_1^2(t)
\end{aligned}
$$



- The pattern should be evident for higher degrees

Geometric view (deCasteljau Algorithm):

- Join the points P_i by line segments
- Join the $t : (1 - t)$ points of those line segments by line segments
- Repeat as necessary
- The $t : (1 - t)$ point on the final line segment is a point on the curve
- The final line segment is tangent to the curve at t



Expanding Terms (Basis Polynomials):

- The original points appear as coefficients of *Bernstein polynomials*

$$
\begin{aligned}
P_0^0(t) &= P_0 1 \\
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_0^2(t) &= (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \\
P_0^3(t) &= (1-t)^3 P_0 + 3(1-t)^2 tP_1 + 3(1-t)t^2 P_2 + t^3 P_3 \\
P_0^n(t) &= \textstyle\sum_{i=0}^n P_i B_i^n(t)
\end{aligned}
$$

where

$$B_i^n(t) = \frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i = \begin{pmatrix} n \\ i \end{pmatrix} (1-t)^{n-i}t^i$$

- The Bernstein polynomials of degree n form a basis for the space of all degree-n polynomials

Recursive evaluation schemes:

- To obtain curve points:
  - Start with given points and form successive, pairwise, affine combinations

$$\begin{aligned} P_i^0 &= P_i \\ P_i^j &= (1-t)P_i^{j-1} + tP_{i+1}^{j-1} \end{aligned}$$

  - The generated points P_i^j are the *deCasteljau points*
- To obtain basis polynomials:
  - Start with 1 and form successive, pairwise, affine combinations

$$\begin{aligned} B_0^0 &= 1 \\ B_i^j &= (1-t)B_i^{j-1} + tB_{i+1}^{j-1} \end{aligned}$$

  where $B_r^s = 0$ when $r < 0$ or $r > s$

Recursive triangle diagrams (upward):

Computing deCasteljau points

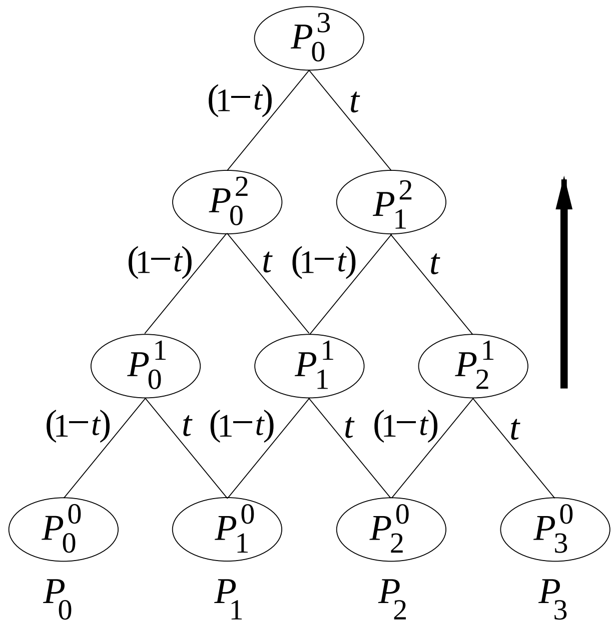
- Each node gets the affine combination of the two nodes entering from below
  - Leaf nodes have the value of their respective points

$$P_1^2 = (1-t)P_1^1 + tP_2^1$$

- Each node gets the sum of the path products entering from below

$$P_1^2 = P_0^1(1-t)(1-t) + P_0^2 t(1-t) + P_0^2(1-t)t + P_3^0 tt$$

$$P_1^2 = (1-t)^2 P_0^1 + 2(1-t)tP_0^2 + t^2 P_3^0$$


P_0^3
(1−t)
t
P_0^2
P_1^2
(1−t)
t
(1−t)
t
P_0^1
P_1^1
P_2^1
(1−t)
t
(1−t)
t
(1−t)
t
P_0^0
P_1^0
P_2^0
P_3^0
P_0
P_1
P_2
P_3

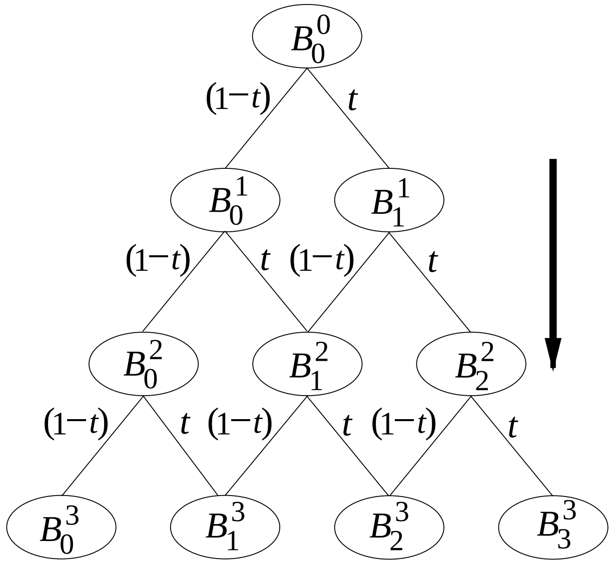
Recursive triangle diagrams (downward):

Computing Bernstein (basis) polynomials

- Each node gets the affine combination of the two nodes entering from above
  - Root node has value 1
  - For other nodes, missing entries above count as zero
- Each node gets the sum of the path products entering from above

  $$B_1^3 = t(1-t)(1-t) + (1-t)t(1-t) + (1-t)t(1-t)t$$

  $$P_1^3 = 3(1-t)^2 t$$


B_0^0
(1−t)
t
B_0^1
B_1^1
(1−t)
t
(1−t)
t
B_0^2
B_1^2
B_2^2
(1−t)
t
(1−t)
t
(1−t)
t
B_0^3
B_1^3
B_2^3
B_3^3

Bernstein Basis Functions

Bernstein Polynomial Properties:

Partition of Unity: $\sum_{i=0}^{n} B_i^n(t) = 1$

Proof:

$$\begin{aligned} 1 &= (t + (1-t))^n \\ &= \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \\ &= \sum_{i=0}^{n} B_i^n(t) \end{aligned}$$

Nonnegativity: $B_i^n(t) \geq 0$, for $t \in [0, 1]$

Proof:

$$\begin{aligned} \binom{n}{i} &> 0 \\ t &\geq 0 \text{ for } 0 \leq t \leq 1 \\ (1-t) &\geq 0 \text{ for } 0 \leq t \leq 1 \end{aligned}$$

Recurrence: $B_0^0(t) = 1$ and $B_i^n(t) = (1-t)B_i^{n-1}(t) + B_{i-1}^{n-1}(t)$

Proof:

$$
\begin{aligned}
B_i^n(t) &= \binom{n}{i} t^i (1-t)^{n-i} \\
&= \binom{n-1}{i} t^i (1-t)^{n-i} + \binom{n-1}{i-1} t^i (1-t)^{n-i} \\
&= (1-t) \binom{n-1}{i} t^i (1-t)^{(n-1)-i} + \\
&\quad t \binom{n-1}{i-1} t^{i-1} (1-t)^{(n-1)-(i-1)} \\
&= (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)
\end{aligned}
$$

Derivatives: $\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$

Proof:

$$\begin{aligned}
\frac{d}{dt}B_i^n(t) &= \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} \\
&= \frac{d}{dt}\frac{i!(n-i)!}{n!}t^i(1-t)^{n-i} \\
&= \frac{i!(n-i)!}{in!}t^{i-1}(1-t)^{n-i} - \\
&\quad \frac{i!(n-i)!}{(n-i)n!}t^i(1-t)^{n-i-1} \\
&= \frac{(i-1)!(n-i)!}{n(n-1)!}t^{i-1}(1-t)^{n-i} - \\
&\quad \frac{i!(n-i-1)!}{n(n-i)!}t^i(1-t)^{n-i-1} \\
&= n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)
\end{aligned}$$

Bézier Splines

Bézier Curve Segments and their Properties

Definition:

- A degree n (order $n+1$) *Bézier curve segment* is

$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t)$$

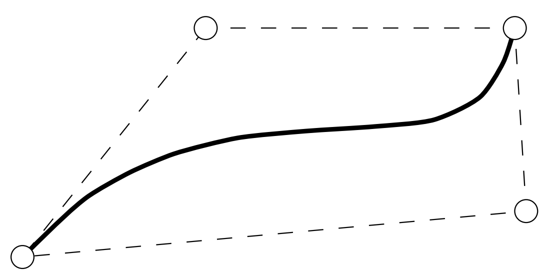
where the P_i are k-dimensional *control points*.

Convex Hull:

$\sum_{i=0}^{n} B_i^n(t) = 1$, $B_i^n(t) \geq 0$ for $t \in [0,1]$

$\Longrightarrow P(t)$ is a convex combination of the P_i for $t \in [0,1]$

$\Longrightarrow P(t)$ lies within convex hull of P_i for $t \in [0,1]$



Affine Invariance:

- A Bézier curve is an affine combination of its control points
- Any affine transformation of a curve is the curve of the transformed control points

$$T\left(\sum_{i=0}^{n} P_i B_i^n(t)\right) = \sum_{i=0}^{n} T(P_i) B_i^n(t)$$

- *This property does not hold for projective transformations!*

Interpolation:

$B_0^n(0) = 1, B_n^n(1) = 1, \sum i = 0^n B_i^n(t) = 1, B_i^n(t) \geq 0$ for $t \in [0, 1]$

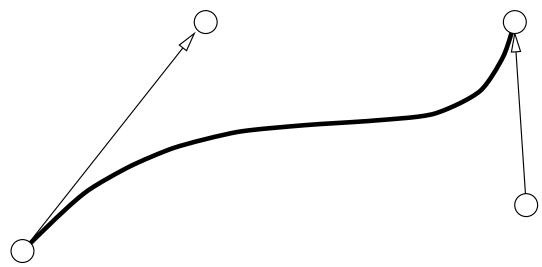
$\Longrightarrow B_i^n(0) = 0$ if $i \neq 0, B_i^n(1) = 0$ if $i \neq n$

$\Longrightarrow P(0) = P_0, P(1) = P_n$

Derivatives:

$$\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

$$\Longrightarrow P'(0) = n(P_1 - P_0),\ P'(1) = n(P_n - P_{n-1})$$

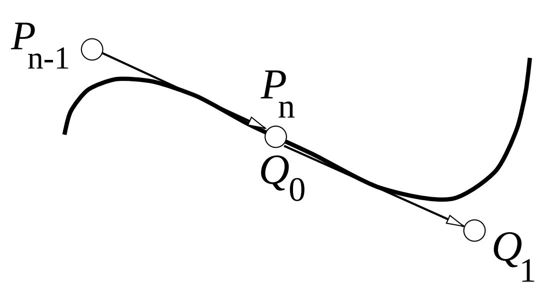


Smoothly Joined Segments (G^1):

- Let P_{n-1}, P_n be the last two control points of one segment
- Let Q_0, Q_1 be the first two control points of the next segment

$$P_n = Q_0$$

$$(P_n - P_{n-1}) = \beta(Q_1 - Q_0) \text{ for some } \beta > 0$$



Recurrence, Subdivision:

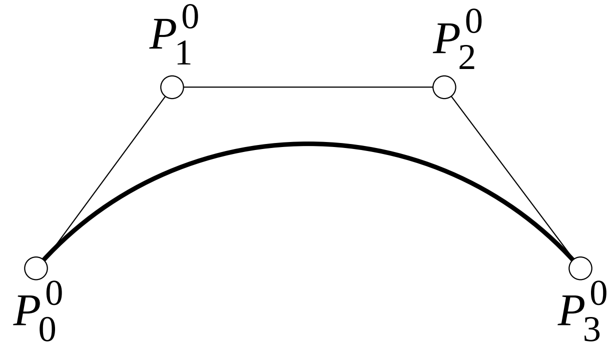
$$B_i^n(t) = (1-t)B_i^{n-1} + tB_{i-1}^{n-1}(t)$$

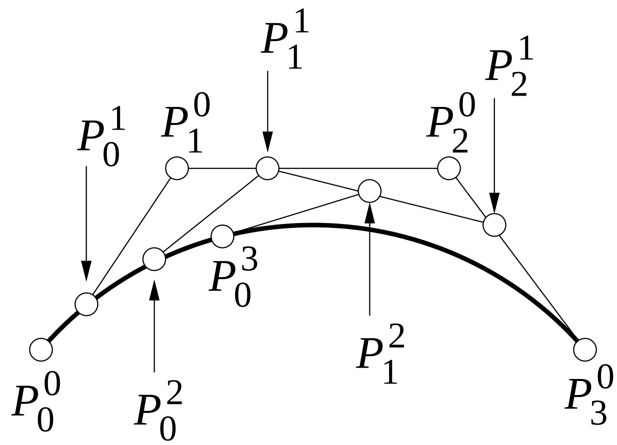
$\Longrightarrow$ deCasteljau's algorithm:

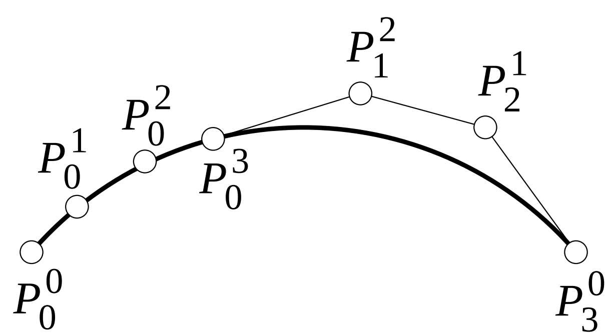
$$\begin{aligned} P(t) &= P_o^n(t) \\ P_i^k(t) &= (1-t)P_i^{k-1}(t) + t)P_{i+1}^{k-1} \\ P_i^0 &= P_i \end{aligned}$$

Use to evaluate point at t, or subdivide into two new curves:

- $P_0^0, P_0^1, \ldots P_0^n$ are the control points for the left half
- $P_n^0, P_{n-1}^1, \ldots P_0^n$ are the control points for the right half


P_1^0
P_2^0
P_0^0
P_3^0


P_1^1
P_2^1
P_0^1
P_1^0
P_2^0
P_0^3
P_1^2
P_0^0
P_0^2
P_3^0


P_1^2
P_2^1
P_0^2
P_0^1
P_0^3
P_0^0
P_3^0

Matrix View:

- Expand each Bernstein polynomial in powers of t
- Represent each expansion as the column of a matrix
- Quadratic example:

$$(1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2$$

$$= (1 - 2t + t^{)} P_0 + (2t - 2t^2)P_0 + (2t - 2t^2)P_1 + t^2 P_2$$

$$= [1 \; t \; t^2] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

- In matrix format:

$$P(t) = T(t)^T M_{BT} P$$

  - $T(t)^T = [1 \; t \; t^2]$ is the *monomial basis*
  - $P_T = M_{BT}P$ is a matrix containing the coefficients of the polynomials for each dimension of $P(t)$
  - M_{BT} is a *change of basis matrix* that converts a specification P of $P(t)$ relative to the Bernstein basis to one relative to the monomial basis

Tensor Product Patches

Tensor Product Patches:

- The *control polygon* is the polygonal mesh with vertices $P_{i,j}$
- The*patch basis functions* are products of curve basis functions

$$P(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} P_{i,j} B_{i,j}^{n}(s,t)$$

where

$$B_{i,j}^{n}(s,t) = B_{i}^{n}(s) B_{j}^{n}(t)$$

Scan in image.

Properties:

- Patch basis functions *sume to one*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} B_i^n(s)B_j^n(t) = 1$$

- Patch basis functions are *nonnegative* on [0, 1] × [0, 1]

$$B_i^n(s)B_j^n(t) \geq 0 \text{ for } 0 \leq s, t \leq 1$$

$\Longrightarrow$ Surface patch is in the *convex hull* of the control points
$\Longrightarrow$ Surface patch is *affinely invariant*
(Transform the patch by transforming the control points)

Subdivision, Recursion, Evaluation:

- As for curves in each variable separately and independently
- *Tangent plane is not produced!*
  - Normals must be computed from partial derivatives

Partial Derivatives:

- Ordinary derivative in each variable separately':

$$\frac{\partial}{\partial s}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}\left[\frac{d}{ds}B_i^n(s)\right]B_j^n(t)$$

$$\frac{\partial}{\partial s}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}B_i^n(s)\left[\frac{d}{dt}B_j^n(t)\right]$$

- Each of the above is a *tangent vector* in a parametric direction
- Surface is *regular* at each (s,t) where these two vectors are linearly independent
- The (unnormalized) *surface normal* is given at any regular point by

$$\pm\left[\frac{\partial}{\partial s}P(s,t)\times\frac{\partial}{\partial t}P(s,t)\right]$$

  (the sign dictates what is the *outward pointing normal*)
- In particular, the *cross-boundary tangent* is given by (e.g., for the $s=0$ boundary):

$$n\sum_{i=0}^{n}\sum_{j=0}^{n}(P_{1,j}-P_{0,j})B_j^n(t)$$

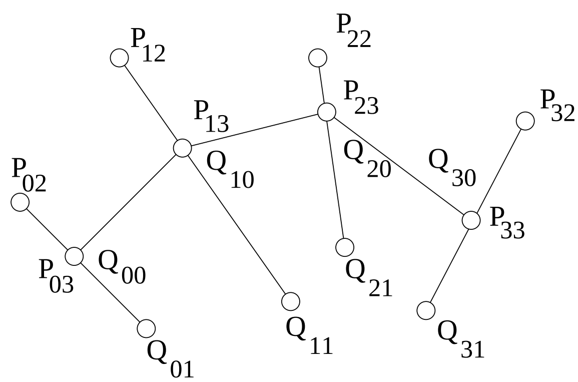
  (and similarly for the other boundaries)

Smoothly Joined Patches:

- Can be achieved by ensuring that

$$(P_{i,n} - P_{i,n-1}) = \beta(Q_{i,1} - Qi, 0) \text{ for } \beta > 0$$

(and correspondingly for other boundaries)



Rendering:

- Divide up into polygons:
  1. By stepping

$$\begin{aligned} s &= 0, \delta, 2\delta, \ldots, 1 \\ t &= 1, \gamma, 2\gamma, \ldots, 1 \end{aligned}$$

     and joining up sides and diagonals to produce a triangular mesh
  2. By subdividing and rendering the control polygon

Barycentric Coordinates (optional)

Coordinate Frames:

- Vector oriented; derived from linear space basis
- One point and n vectors in space of dimension: n : $D_n, \vec{v}_0, \ldots, \vec{v}_{n-1}$
  - Vectors $\vec{v}_i$ are *linearly independent*

Barycentric Frames:

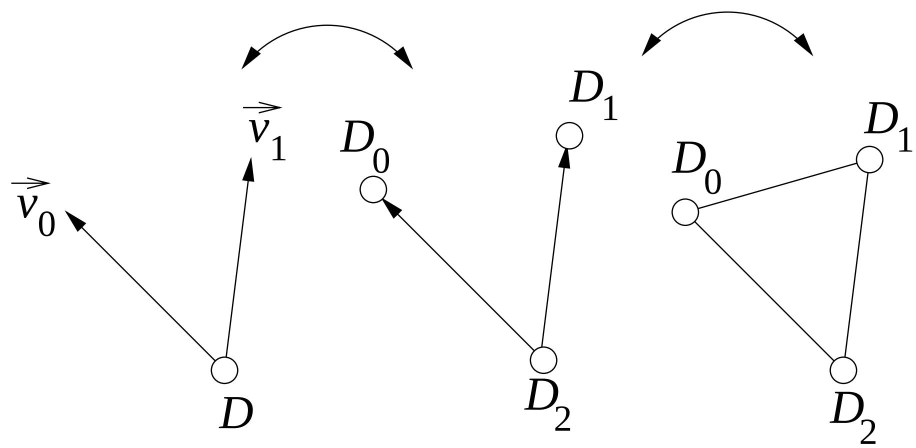
- Point oriented
- $n + 1$ points in space of dimension N : $D_0, \ldots, D_n$
  - Points are in *general position*

Frames of Both Types Are Equivalent

- Express each $\vec{v}_i$ as $D_i - D_n$ for $D_i = D_n + v_i$

$$\begin{aligned}
P &= D_n + \sum_{i=0}^{n-1} p_i \vec{v}_i \\
&= D_n + \sum_{i=0}^{n-1} p_i (D_i - D_n) \\
&= (1 - \sum_{i=0}^{n-1} p_i) D_n + \sum_{i=0}^{n-1} p_i D_i \\
&= \sum_{i=0}^{n-1} w_i D_i \text{ where } \sum_{i=0}^{n-1} w_i = 1
\end{aligned}$$

- And, of course, conversely


$\vec{v}_0$
$\vec{v}_1$
D
D_0
D_1
D_2
D_0
D_1
D_2

Triangular Patches (optional)

deCasteljau Revisited Barycentrically:

- Linear blend expressed in barycentric terms

$$(1-t)P_0 + tP_1 = rP_0 + tP_1 \text{ where } r + t = 1$$

- Higher powers and a symmetric form of the Bernstein polynomials:

$$\begin{aligned}
P(t) &= \sum_{i=0}^{n} P_i \left(\frac{n!}{i!(n-i)!}\right)(1-t)^{n-i}t^i \\
&= \sum_{\substack{i+j=n \\ i \geq 0, j \geq 0}} P_i \left(\frac{n!}{i!j!}\right) t^i r^j \text{ where } r + t = 1 \\
\Longrightarrow & \sum_{\substack{i+j=n \\ i \geq 0, j \geq 0}} P_{ij} B^n_{ij}(r,t)
\end{aligned}$$

- Examples

$$\begin{aligned} B^0_{00}(r,t) &= 1 \\ B^1_{01}(r,t), B^1_{10}(r,t) &= r, t \\ B^2_{02}(r,t), B^2_{11}(r,t), B^2_{20}(r,t) &= r^2, 2rt, t^2 \\ B^3_{03}(r,t), B^3_{12}(r,t), B^3_{21}(r,t), B^3_{30}(r,t) &= r^3, 3r^2t, 3rt^2, t^3 \end{aligned}$$

Surfaces – Barycentric Blends on Triangles:

- Formulas

$$\begin{aligned} P(r,s,t) &= \sum_{\substack{i+j+k=n \\ i\geq 0, j\geq 0, k\geq 0}} P_{ijk} B^n_{ijk}(r,s,t) \\ B^n_{ijk}(r,s,t) &= \frac{n!}{i!j!k!} r^i s^j t^k \end{aligned}$$

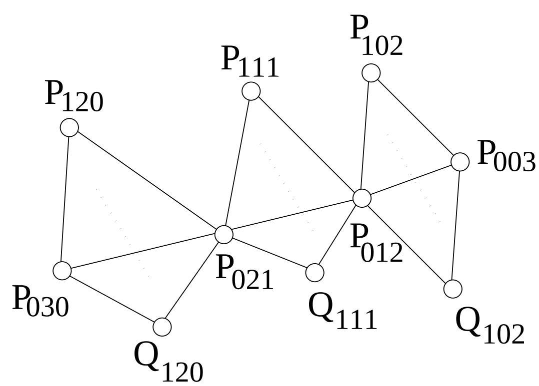
Triangular deCasteljau:

- Join adjacently indexed P_{ijk} by triangles
- Find $r : s : t$ barycentric point in each triangle
- Join adjacent points by triangles
- Repeat
  - Final point is the surface point $P(r, s, t)$
  - final triangle is tangent to the surface at $P(r, s, t)$
- Triangle up/down schemes become tretrahedral up/down schemes

Scan in image.

Properties:

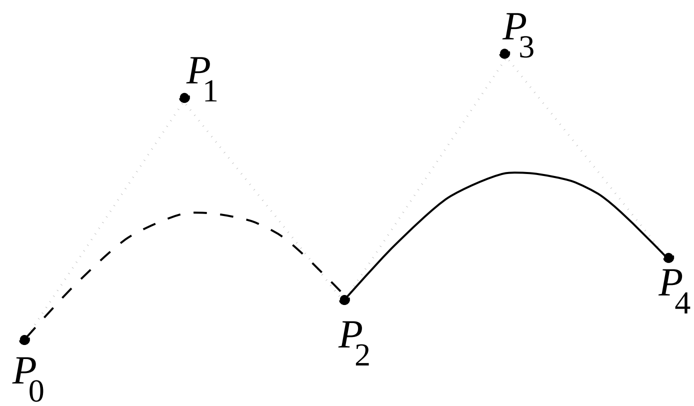
- Each boundary curve is a Bézier curve
- Patches will be joined smoothly if pairs of boundary triangles are planar as shown



Discontinuities in Splines

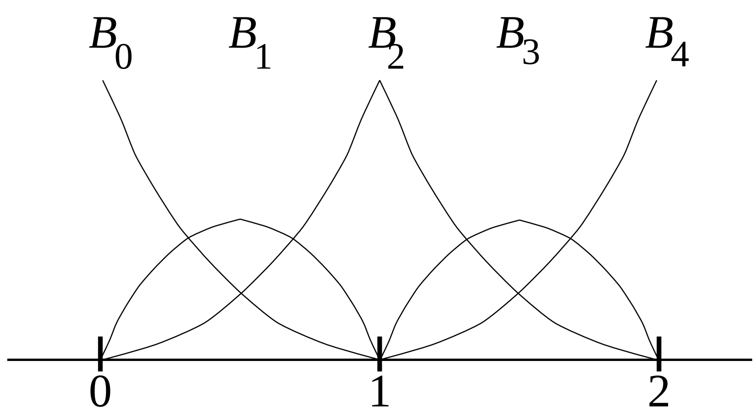
Bézier Discontinuities:

- Two Bézier segments can be completely disjoint
- Two segments join if they share last/first control point



Common Parameterization and Blending Functions

- Joined curves can be given common parameterization
  - Parameterize first segment with $0 \leq t < 1$
  - Parameterize nest segment with $1 \leq t \leq 2$, etc.
- Look at blending/basis polynomials under this parameterization
  - Combine those for common P_j into a single piecewise polynomial



Combined Curve Segments

- Curve is $P(t) = P_0B_0(t) + P_1B_1(t) + P_2B_2(t) + P_3B_3(t) + P_4B_4(t)$, where

$$B_0(t) = \begin{cases} (1-t)^2 & 0 \le t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

$$B_1(t) = \begin{cases} 2((1-t)t & 0 \le t < 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

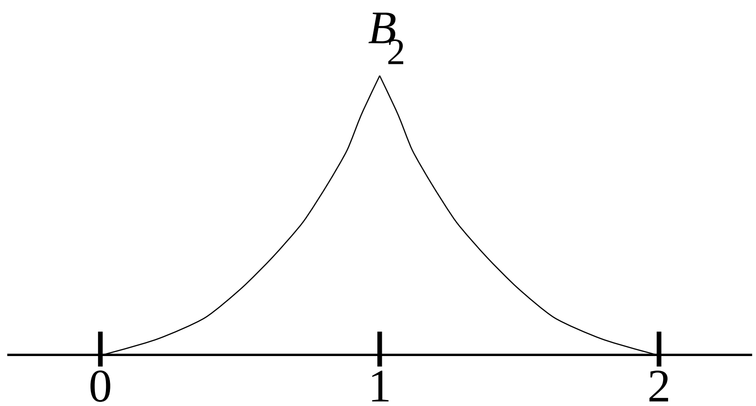
$$B_2(t) = \begin{cases} t^2 & 0 \le t < 1 \\ (2-t)^2 & 1 \le t \le 2 \end{cases}$$

$$B_3(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2(2-t)(t-1) & 1 \le t \le 2 \end{cases}$$

$$B_4(t) = \begin{cases} 0 & 0 \le t < 1 \\ (t-1)^2 & 1 \le t \le 2 \end{cases}$$

Curve Discontinuities from Basis Discontinuities

- P_2 is scaled by $B_2(t)$, which has a discontinuous derivative
- The corner in the curve results from this discontinuity



Spline Continuity

Smoother Blending Functions:

- Can $B_0(t), \ldots, B_4(t)$ be replaced by smoother functions?
  - Piecewise polynomials on $0 \leq t \leq 2$
  - Continuous derivatives
- Yes, but we lose one degree of freedom
  - Curve has no corner if segments share a common tangent
  - Tangent is given by the chords $\overline{P_1 P_2}$, $\overline{P_2 P_3}$
  - An equation constrains P_1, P_2, P_3

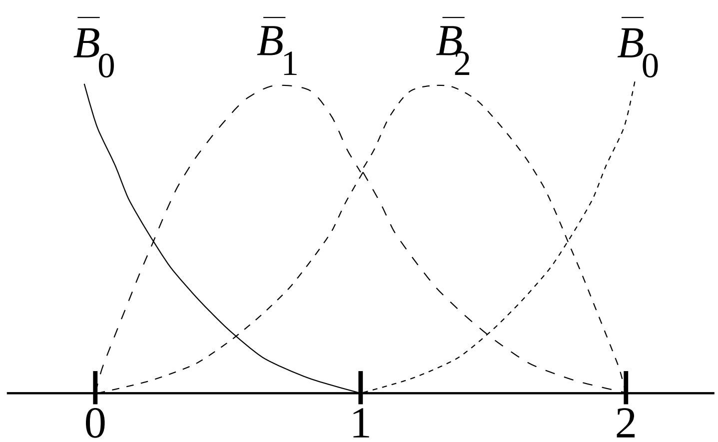
    $P_3 - P_2 = P_2 - P_1 \Longrightarrow P_2 = \frac{P_1 + P_3}{2}$
- This equation leads to combinations:

$$P_0 B_0(t) \;+\; P_1 \left(B_1(t) + \tfrac{1}{2} B_2(t)\right) \;+\; P_3 \left(\tfrac{1}{2} B_2(t) + B_3(t)\right) \;+\; P_4 B_4(t)$$

Spline Basis:

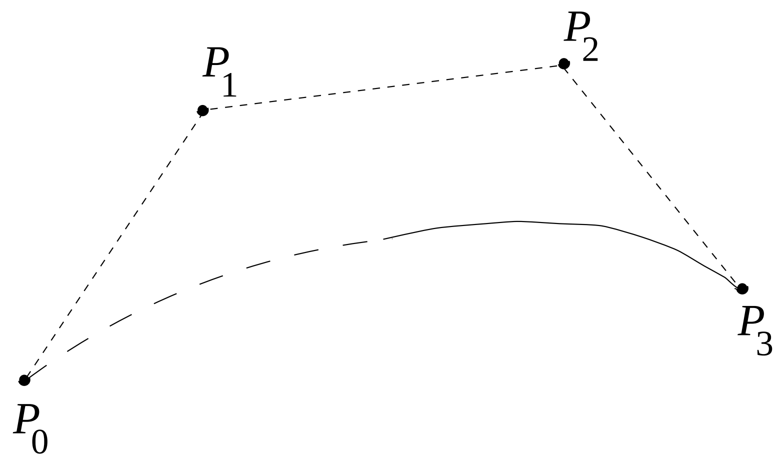
- Combined functions form a smoother *spline basis*

$$\begin{aligned}
\overline{B}_0(t) &= B_0(t) \\
\overline{B}_1(t) &= \left(B_1(t) + \frac{1}{2}B_2(t)\right) \\
\overline{B}_2(t) &= \left(\frac{1}{2}B_2(t) + B_3(t)\right) \\
\overline{B}3_(t) &= B_4(t)
\end{aligned}$$



Smoother Curves:

- Control points used with this basis produce smoother curves.

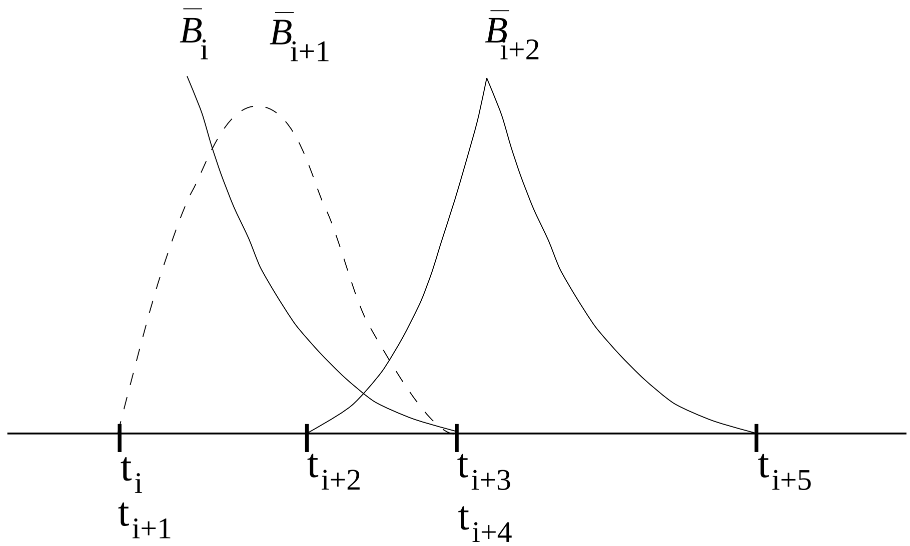


B-Splines

General B-Splines:

- Nonuniform B-splines (NUBS) generalize this construction
- A B-spline, $B_i^d(t)$, is a piecewise polynomial:
  - each of its segments is of degree $\leq d$
  - it is defined for all t
  - its segmentation is given by *knots* $t = t_0 \leq t_1 \leq \cdots \leq t_N$
  - it is zero for $T < T_i$ and $T > T_{i+d+1}$
  - it may have a discontinuity in its $d - k + 1$ derivative at $t_j \in \{t_i, \ldots, t_{i+d+1}\}$, if t_j has multiplicity k
  - it is nonnegative for $t_i < t < t_{i+d+1}$
  - $B_i^d(t) + \cdots + B_{i+d}(t) = 1$ for $t_{i+d} \leq t < t_{i+d+1}$, and all other $B_j^d(t)$ are zero on this interval
  - Bézier blending functions are the special case where all knots have multiplicity $d + 1$

Example (Quadratic):



Evaluation:

- There is an efficient, recursive evaluation scheme for any curve point
- It generalizes the triangle scheme (deCasteljau) for Bézier curves
- Example (for cubics and $t_{i+3} \leq t < t_{i+4}$):