Computing Intersections

- Ray Intersection Computations
  - Planes
  - Polygons
  - Triangles
  - Convex Polyhedra (e.g. Boxes)
  - Quadrics

Ray Plane Intersection

This is even easier than a sphere

- Ray Equation (always parametric) $\mathbf{r}(t) = \mathbf{o} + t\vec{d}$
- Plane equation (implicit) $Ax + By + Cz + D = 0$
- Intersection equation $A(o_x + td_x) + B(o_y + td_y) + C(o_z + td_z) + D = 0$
- Rearranging terms gives

$at + b = 0$ where

$$a = Ad_x + Bd_y + Cd_z = [A\ B\ C\ 0] \cdot \vec{d}$$

$$b = Ao_x + Bo_y + Co_z + D = [A\ B\ C\ D] \cdot \mathbf{o}$$

- So then $t = -b/a$

- Have to check for $a = 0$, which means that the ray is parallel to the plane (ray direction orthogonal to plane normal) and there's no intersection

- Sign of a indicates whether ray is coming from front or back of plane

- Sign of b indicates which side of plane the ray origin is on

Ray Polygon Intersection

Unfortunately, a plane is not a polygon

- If $t \geq 0$ we have a valid intersection with the plane

- Now we need to check whether this is within the polgon or not

- This can be done by computing the *winding number* for the point and polygon

Ray Triangle Intersection

Triangles are harder than planes but easier than general polygons

- Ray Equation (always parametric) $\mathbf{r}(t) = \mathbf{o} + t\vec{d}$
- Triangle equation (barycentric - two parameters) $\mathbf{p}(a, b) = u\mathbf{p}_0 + v\mathbf{p}_1 + (1 - u - v)\mathbf{p}_2$
- Intersection equation(s) $\mathbf{o} + t\vec{d} = u\mathbf{p}_0 + v\mathbf{p}_1 + (1 - u - v)\mathbf{p}_2$
- Three equations (for x, y, z) in three unknowns - t, u, v

$$\begin{bmatrix} \vec{d} & \mathbf{p}_2 - \mathbf{p}_0 & \mathbf{p}_2 - \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{p}_2 - \mathbf{o} \end{bmatrix}$$

$$\begin{bmatrix} d_x & x_2 - x_0 & x_2 - x_1 \\ d_y & y_2 - y_0 & y_2 - y_1 \\ d_z & z_2 - z_0 & z_2 - z_1 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} = \begin{bmatrix} x_2 - o_x \\ y_2 - o_y \\ z_2 - o_z \end{bmatrix}$$

- This can be solved using Cramer's rule

$$\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \frac{1}{\det \begin{bmatrix} \vec{d} & \mathbf{p}_2 - \mathbf{p}_0 & \mathbf{p}_2 - \mathbf{p}_1 \end{bmatrix}} \begin{bmatrix} \det \begin{bmatrix} \mathbf{p}_2 - \mathbf{o} & \mathbf{p}_2 - \mathbf{p}_0 & \mathbf{p}_2 - \mathbf{p}_1 \end{bmatrix} \\ \det \begin{bmatrix} \vec{d} & \mathbf{p}_2 - \mathbf{o} & \mathbf{p}_2 - \mathbf{p}_1 \end{bmatrix} \\ \det \begin{bmatrix} \vec{d} & \mathbf{p}_2 - \mathbf{p}_0 & \mathbf{p}_2 - \mathbf{o} \end{bmatrix} \end{bmatrix}$$

- These determinants can be expressed in terms of dot and cross products and optimized by identifying common subexpressions

$$\vec{c}_1 = [\mathbf{p}_2 - \mathbf{o}] \times [\mathbf{p}_2 - \mathbf{p}_0]$$

$$\vec{c}_2 = \vec{d} \times [\mathbf{p}_2 - \mathbf{p}_1]$$

so

$$\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \frac{1}{\vec{c}_2 \cdot [\mathbf{p}_2 - \mathbf{p}_0]} \begin{bmatrix} \vec{c}_1 \cdot [\mathbf{p}_2 - \mathbf{p}_1] \\ \vec{c}_2 \cdot [\mathbf{p}_2 - \mathbf{o}] \\ \vec{c}_1 \cdot \vec{d} \end{bmatrix}$$

Ray Polyhedron Intersection

Multiple bounding planes of polyhedra are treated in parallel

- Start by computing a value of t for the intersection of the ray with each of the planes bounding the (convex) polyhedron.
- Classify each such intersection as "entering" or "leaving" using the sign of a in the expression for the ray plane intersection as given earlier
- Identify the "farthest entering" value for the ray, the largest value of t classified as "entering" and the "nearest leaving" value, the smallest value of t classified as "leaving"
- If the "farthest entering" value is less than or equal to the "nearest leaving" value, then the "farthest entering" value is the nearest intersection and the "nearest leaving" value is the farthest intersection
- Otherwise the ray misses the polyhedron
- This can be optimized for early termination by updating and testing the t values incrementally as they are generated and stopping if it is clear that the ray has missed the polyhedron

Intersection with General Quadrics

This is more general and more costly than spheres

- Begin with general quadric implicit form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

- And the usual parametric ray equation $\mathbf{r}(t) = \mathbf{o} + t\vec{d}$

- Substitute to get

$$A(o_x + td_x)^2 + B(o_y + td_y)^2 + C(o_z + td_z)^2 +$$
$$D(o_x + td_x)(o_y + td_y) + E(o_x + td_x)(o_z + td_z) + F(o_y + td_y)(o_z + td_z) +$$
$$G(o_x + td_x) + H(o_y + td_y) + I(o_z + td_z) + J = 0$$

- Then rearrange to get $at^2 + bt + c = 0$ where

$$a = Ad_x^2 + Bd_y^2 + Cd_z^2 + Dd_xd_y + Ed_xd_z + Fd_yd_z$$

$$\begin{aligned} b = {} & 2Ao_xd_x + 2Bo_yd_y + 2Co_zd_z + \\ & Do_xd_y + Do_yd_x + Eo_xd_z + Eo_zd_x + Fo_yd_z + Fo_zd_y + \\ & Gd_x + Hd_y + Id_z \end{aligned}$$

$$c = Ao_x^2 + Bo_y^2 + Co_z^2 + Do_xo_y + Eo_xo_z + Fo_yo_z + Go_x + Ho_y + Io_z + J$$

- Solve this using the quadratic formula, as for the sphere
  - $D = b^2 - 4ac$
  - If $D < 0$ the ray misses the object
  - If $D = 0$ the ray may hit the object where $t = -b/2a$
  - If $D > 0$ the ray may hit the object at two points, where $t_0 = (-b - \sqrt{D})/2a$ and $t_1 = (-b + \sqrt{D})/2a$

- As with the sphere, check these values of t against t_{min} and t_{max} for a valid intersection

- If only partial surfaces are to be used, then we will also have to check for this, again as with the sphere