Conic Sections, Quadrics, Ray Tracing Spheres

- Conics
  - Implicit form
  - Parametric form
- Quadrics
  - Implicit form
  - Parametric form
- Ray-Sphere Intersection
  - Computing the intersection
  - Differential Geometry

Conic Curves

Conic Sections (Implicit form)

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Parabola

$$y^2 = 4ax \quad a > 0$$

Conic Sections (Parametric form)

- Ellipse

$$\begin{aligned} x(t) &= a\frac{1-t^2}{1+t^2} \\ y(t) &= b\frac{2t}{1+t^2} \quad (-\infty < t < +\infty) \end{aligned}$$

- Hyperbola

$$\begin{aligned} x(t) &= a\frac{1+t^2}{1-t^2} \\ y(t) &= b\frac{2t}{1-t^2} \quad (-\infty < t < +\infty) \end{aligned}$$

- Parabola

$$
\begin{aligned}
x(t) &= at^2 \\
y(t) &= 2at \;\; (-\infty < t < +\infty)
\end{aligned}
$$

Quadric Surfaces

Surface Equivalent of Conic Sections (Implicit Form)

- 6 types of Quadric Surfaces
  - Ellipsoid $F(x, y, z) = (\frac{x^2}{a}) + (\frac{y^2}{b}) + (\frac{z^2}{c}) - 1 = 0$
  - Alternatively, apply affine transforms to the generic sphere $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$
  - Elliptic Cone $F(x, y, z) = x^2 + y^2 - z^2 = 0$
  - Hyperboloid of One Sheet $F(x, y, z) = x^2 + y^2 - z^2 - 1 = 0$
  - Hyperboloid of Two Sheets $F(x, y, z) = x^2 - y^2 - z^2 - 1 = 0$
  - Elliptic Paraboloid $F(x, y, z) = x^2 + y^2 - z = 0$
  - Hyperbolic Paraboloid $F(x, y, z) = -x^2 + y^2 - z = 0$

Quadric Surfaces

Surface Equivalent of Conic Sections (Parametric Form)

- Ellipsoid $F(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$,
  $\theta \in (0, \pi), \phi \in (0, 2\pi)$
- Elliptic Cone $F(u, v) = (v\cos(u), v\sin(u), v)$,
  $u \in (-\pi, \pi), v \in \mathcal{R}$
- Hyperboloid of One Sheet $F(u, v) = (\sec(v)\cos(u), \sec(v)\sin(u), \tan(v))$,
  $u \in (-\pi, \pi), v \in (-\pi/2, pi/2)$
- Hyperboloid of Two Sheets $F(u, v) = (\sec(v)\cos(u), \sec(v)\tan(u), \tan(v))$,
  $u \in (-\pi, \pi), v \in (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$
- Elliptic Paraboloid $F(u, v) = (v\cos(u), v\sin(u), v^2)$,
  $u \in (-\pi, \pi), v \geq 0$
- Hyperbolic Paraboloid $F(u, v) = (v\tan(u), v\sec(u), v^2)$,
  $u \in (-\pi, \pi), v \geq 0$

Ray-Sphere Intersection

Generalizes to all Ellipsoids

- Ray Equation (always parametric) $\mathbf{r}(t) = \mathbf{o} + t\vec{d}$

- Sphere equation (implicit) $x^2 + y^2 + z^2 - 1 = 0$

- Intersection equation $(\mathbf{o}_x + t\vec{d}_x)^2 + (\mathbf{o}_y + t\vec{d}_y)^2 + (\mathbf{o}_z + t\vec{d}_z)^2 = 1$

- Rearranging terms gives

$At^2 + bt + C = 0$ where

$$A = \vec{d}_x^2 + \vec{d}_y^2 + \vec{d}_z^2 = \vec{d} \cdot \vec{d}$$

$$b = 2(\vec{d}_x \mathbf{o}_x + \vec{d}_y \mathbf{o}_y + \vec{d}_z \mathbf{o}_z) = 2(\vec{d} \cdot \vec{o}) = 2B$$

$$C = \mathbf{o}_x^2 + \mathbf{o}_y^2 + \mathbf{o}_z^2 - 1 = \vec{o} \cdot \vec{o} - 1$$

and

$$\vec{o} = \mathbf{o} - \mathcal{O}$$

Ray-Sphere Intersection

- Quadratic form gives 0, 1 or 2 solutions, depending on value of $D = B^2 - AC$
- If $D < 0$, no real solutions, ray misses sphere
- If $D = 0$, 1 real solution, ray might be tangent to sphere at a point $t = -B/A$
- If $D > 0$, 2 real solutions, ray might hit sphere $t_0 = (-B + \sqrt{D})/A$ and $t_1 = (-B - \sqrt{D})/A$
- Note that if you use a unit vector for ray direction, $A = 1$
- Also note that the sign of AC indicates whether the ray origin is inside (negative), on (zero) or outside (positive) the circle
- How to determine intersection? It's the smallest value of t between t_{min} and t_{max} if it exists
- For partial spheres, also need to check ϕ, θ and z against 0 and ϕ_{max}, 0 and θ_{max}, and z_{min} and z_{max}, respectively
- Since $\frac{x}{y} = \frac{\sin(\theta)\sin(\phi)}{\sin(\theta)\cos(\phi)} = \tan(\phi)$, $\phi = \arctan(\frac{y}{x})$
- Similarly since $z = \cos(\theta)$, $\theta = \arccos(z)$

- To generalize, transform ray to sphere space before calculating, then transform results back to world space

Ray-Sphere Intersection

Sphere Differential Geometry

- To make u, v range from 0 to 1, we have $u = \frac{\phi}{\phi_{max}}$ and $v = \frac{\theta - \theta_{min}}{\theta_{max} - \theta_{min}}$
- Given that $x = \sin(\theta)\cos(\phi)$,
  $\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(\sin(\theta)\cos(\phi)$
  $= \sin(\theta)\frac{\partial}{\partial u}(\cos(\phi))$
  $= \sin(\theta)(-\phi_{max}\sin(\phi))$
  $= -\phi_{max} y$

- Similar algebra gives $\frac{\partial y}{\partial u} = \phi_{max} x$ and $\frac{\partial z}{\partial u} = 0$
- Using similar algebra to get $\frac{\partial \mathbf{p}}{\partial v}$, we end up with

$$\frac{\partial \mathbf{p}}{\partial u} = (-\phi_{max} y, \phi_{max} x, 0)$$

and

$$\frac{\partial \mathbf{p}}{\partial v} = (\theta_{max} - \theta_{min})(z\cos(\phi), z\sin(\phi), -\sin(\theta))$$

- To get a normal at the intersection point $\mathbf{p}$, either compute $\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}$ and normalize, or, since this is a unit sphere at the origin, compute the vector $\vec{p} = \mathbf{p} - \mathcal{O}$, i.e. the coordinates of the intersection point are the coordinates of the normal vector
- To complete the local coordinate system, let $\vec{S} = \widehat{\frac{\partial \mathbf{p}}{\partial u}}$ ($\hat{}$ (hat) means normalized to unit length) and $\vec{T} = \vec{N} \times \vec{S}$