Geometric Spaces and Operations

Mathematical underpinnings of computer graphics

- Hierarchy of geometric spaces
  - Vector spaces
  - Affine spaces
  - Euclidean spaces
  - Cartesian spaces
  - Projective spaces
- Affine geometry and transformations
- Projective transformations and perspective
- Matrix formulations of transformations
- Viewing transformations
- Quaternions and surface orientation

Formally, a space is defined by

- A set of objects
- Operations on the objects
- Axioms defining invariant properties

Vector Spaces

Definition:

- Set of *vectors* $\mathcal{V}$
- Operations on $\vec{u},\ \vec{v} \in \mathcal{V}$:
  - *Addition:* $\vec{u} + \vec{v} \in \mathcal{V}$
  - *Scalar Multiplication:* $\alpha\vec{u} \in \mathcal{V}$ where $\alpha \in$ some field $\mathcal{F}$
- *Axioms*
  - *Unique zero element:* $0 + \vec{u} = \vec{u}$
  - *Field unit element:* $1\vec{u} = \vec{u}$
  - *Addition commutative:* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - *Addition associative:* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  - *Distributive scalar multiplication:* $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
- Additional definitions
  - Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.
  - Then $\mathcal{B}$ *spans* $\mathcal{V}$ iff any $\vec{v} \in \mathcal{V}$ can be written as $\vec{v} = \sum_{i=1}^{n} \alpha_i\vec{v}_i$.
  - $\sum_{i=1}^{n} \alpha_i\vec{v}_i$ is called a *linear combination* of the vectors in $\mathcal{B}$.
  - $\mathcal{B}$ is called a *basis* of $\mathcal{V}$ if it is a minimal spanning set.
  - All bases of $\mathcal{V}$ contain the same number of vectors.

  - The number of vectors in any basis of $\mathcal{V}$ is called the *dimension* of $\mathcal{V}$.
- Comments:
  - We are interested in 2 and 3 dimensional spaces.
  - No definition of distance (size) exists yet.
  - Angles and points have not been defined.

Affine Spaces

Definition:

- A set of vectors $\mathcal{V}$ and a set of *points* $\mathcal{P}$
- $\mathcal{V}$ is a vector space.
- Point-vector sum: $P + \vec{v} = Q$ with $P,\ Q \in \mathcal{P}$ and $\vec{v} \in \mathcal{V}$
- Point subtraction: For $P, Q \in \mathcal{P}$ and $\vec{v} \in \mathcal{V}$, if $P + \vec{v} = Q$, then $Q - P \equiv \vec{v}$
- Additional definitions:
  - A *frame* $F = (\mathcal{B}, \mathcal{O})$ where $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis of $\mathcal{V}$ and the point $\mathcal{O}$ is called the *origin* of the frame.
  - The dimension of F is the same as the dimension of $\mathcal{V}$.
- Comments:
  - Still no distances or angles
  - Closer to what we want for graphics
  - The space has no distinguished origin

Euclidean Spaces

Definition:

- A *metric space* is any space with a *distance metric* $d(P, Q)$ defined on its elements.
- Distance metric axioms:
  - $d(P, Q) \geq 0$
  - $d(P, Q) = 0$ iff $P = Q$
  - $d(P, Q) = d(Q, P)$
  - $d(P, Q) \leq d(P, R) + d(R, Q)$ (triangle inequality)
- *Euclidean* distance metric:

$$d^2(P, Q) = (P - Q) \cdot (P - Q)$$

- Comments:
  - Euclidean metric based on dot product
  - Dot product defined on vectors
  - Distance metric defined on points
  - Distance is a property of the space, not a frame
- Dot product axioms:

  - $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
  - $\alpha(\vec{u} \cdot \vec{v}) = (\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Additional definitions:
  - The *norm* of a vector $\vec{u}$ is given by $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$.
  - Angles are defined by their cosines: $\cos(\angle\vec{u}\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$
  - Orthogonal vectors: $\vec{u} \cdot \vec{v} = 0 \rightarrow \vec{u} \perp \vec{v}$

Cartesian Spaces

Definition:

- A frame $(\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$ is *orthonormal* iff
  - $\vec{i}$, $\vec{j}$, and $\vec{k}$ are *orthogonal*, i.e. $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ and
  - $\vec{i}$, $\vec{j}$, and $\vec{k}$ are *normal*, i.e. $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$
- Additional definitions:
  - The *standard frame* $F_s = (\vec{i}, \vec{j}, \vec{k}, \mathcal{O})$
  - Points can be distinguished from vectors using an extra coordinate
    * 0 for vectors: $\vec{v} = (v_x, v_y, v_z, 0)$ means $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$
    * 1 for points: $P = (p_x, p_y, p_z, 1)$ means $P = p_x\vec{i} + p_y\vec{j} + p_z\vec{k} + \mathcal{O}$
    * This is known as a *homogeneous* representation
- Comments
  - Coordinates have no meaning without an associated frame
  - There will be other ways to look at the extra coordinate
  - Sometimes we are sloppy and omit the extra coordinate
  - Assume standard frame unless specified otherwise
  - Points and vectors are different
  - Points and vectors have different operations
  - Points and vectors transform differently

Linear Transformations

Vector space $\mathcal{V}$

- Linear combinations of vectors in $\mathcal{V}$ are in $\mathcal{V}$
- For $\vec{u}, \vec{v} \in \mathcal{V}$
  - $\vec{u} + \vec{v} \in \mathcal{V}$
  - $\alpha\vec{u} \in \mathcal{V}$ for any scalar α
  - In general, $\sum_i \alpha_i \vec{u}_i \in \mathcal{V}$ for any scalars α_i
- Linear transformations
  - Let $\mathbf{T} : \mathcal{V}_0 \mapsto \mathcal{V}_1$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are vector spaces
  - Then $\mathbf{T}$ is *linear* iff
    * $\mathbf{T}(\vec{u} + \vec{v}) = \mathbf{T}(\vec{u}) + \mathbf{T}(\vec{v})$
    * $\mathbf{T}(\alpha\vec{u}) = \alpha\mathbf{T}(\vec{u})$
    * In general, $\mathbf{T}\left(\sum_i \alpha_i \vec{u}_i\right) = \sum_i \alpha_i \mathbf{T}\left(\vec{u}_i\right)$

Affine Transformations

Affine space $\mathcal{A} = (\mathcal{V}, \mathcal{P})$

- Recall that for $\vec{u} \in \mathcal{V}$ and $P \in \mathcal{P}$, $P + \vec{u} \in \mathcal{P}$
- Define *point blending*:
  - For $P, P_1, P_2 \in \mathcal{P}$ and scalar α, if $P = P_1 + \alpha\,(P_2 - P_1)$ then $P \equiv (1 - \alpha)\,P_1 + \alpha P_2$
  - This can also be written $P \equiv \alpha_1 P_1 + \alpha_2 P_2$ where $\alpha_1 + \alpha_2 = 1$
  - Geometrically, $\frac{|P-P_0|}{|P-P_1|} = \frac{d_1}{d_2}$ or $P = \frac{d_1 P_1 + d_2 P_2}{d_1 + d_2}$
  - In general, $\sum_i \alpha_i P_i$ is a *point* iff $\sum_i \alpha_i = 1$
- Vectors can always be combined linearly $\sum_i \alpha_i \vec{u}_i$
- Given point subtraction, $\sum_i \alpha_i P_i$ is a *vector* iff $\sum_i \alpha_i = 0$
- Points can be combined linearly $\sum_i \alpha_i P_i$ iff
  - The coefficients sum to 1, giving a point ("affine combination")
  - The coefficients sum to 0, giving a vector ("vector combination")
  - Example affine combination:

$$P(t) = P_0 + t(P_1 - P_0) = (1 - t)P_0 + tP_1$$

  - This says any point on the line is an affine combination of the line segment's endpoints.
- Affine transformations
  - Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ where $\mathcal{A}_0$ and $\mathcal{A}_1$ are affine spaces
  - $\mathbf{T}$ is said to be an *affine transformation* iff
    - $\mathbf{T}$ maps vectors to vectors and points to points
    - $\mathbf{T}$ is a linear transformation on the vectors
    - $\mathbf{T}(P + \vec{u}) = \mathbf{T}(P) + \mathbf{T}(\vec{u})$
  - Properties of affine transformations
    - $\mathbf{T}$ preserves affine combinations:

$$\mathbf{T}(\alpha_0 P_0 + \cdots + \alpha_n P_n) = \alpha_0 \mathbf{T}(P_0) + \cdots + \alpha_n \mathbf{T}(P_n)$$

where $\sum_i \alpha_i = 0$ or $\sum_i \alpha_i = 1$

    - $\mathbf{T}$ maps lines to lines:

$$\mathbf{T}((1-t)P_0 + tP_1) = (1-t)\mathbf{T}(P_0) + t\mathbf{T}(P_1)$$

    - $\mathbf{T}$ is affine iff it preserves ratios of distance along a line:

$$P = \frac{d_0 P_0 + d_1 P_1}{d_0 + d_1} \Rightarrow \mathbf{T}(P) = \frac{d_0 \mathbf{T}(P_0) + d_1 \mathbf{T}(P_1)}{d_0 + d_1}$$

  * $\mathbf{T}$ maps parallel lines to parallel lines (can you prove this?)
- Example affine transformations
  * Rigid body motions (translations, rotations)
  * Scales, reflections
  * Shears

Matrix Representation of Transformations

- Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be affine spaces.
  Let $\mathbf{T} : \mathcal{A}_0 \mapsto \mathcal{A}_1$ be an affine transformation.
  Let $F_0 = (\vec{i}_0, \vec{j}_0, \mathcal{O}_0)$ be a frame for $\mathcal{A}_0$.
  Let $F_1 = (\vec{i}_1, \vec{j}_1, \mathcal{O}_1)$ be a frame for $\mathcal{A}_1$.
- Let $P = x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0$ be a point in $\mathcal{A}_0$.
  The *coordinates* of P relative to $\mathcal{A}_0$ are $(x, y, 1)$.

  This can also be represented in vector form as $P = \begin{bmatrix} \vec{i}_0 & \vec{j}_0 & \mathcal{O}_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- What are the coordinates $(x', y', 1)$ of $\mathbf{T}(P)$ relative to F_1?
  - An affine transformation is characterized by the image of a frame in the domain.

$$\begin{aligned} \mathbf{T}(P) &= \mathbf{T}(x\vec{i}_0 + y\vec{j}_0 + \mathcal{O}_0) \\ &= x\mathbf{T}(\vec{i}_0) + y\mathbf{T}(\vec{j}_0) + \mathbf{T}(\mathcal{O}_0) \end{aligned}$$

  - $\mathbf{T}(\vec{i}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{i}_0) = t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1$.
  - Likewise $\mathbf{T}(\vec{j}_0)$ must be a linear combination of $\vec{i}_1$ and $\vec{j}_1$, say $\mathbf{T}(\vec{j}_0) = t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1$.
  - Finally $\mathbf{T}(\mathcal{O}_0)$ must be an affine combination of $\vec{i}_1$, $\vec{j}_1$, and $\mathcal{O}_1$, say $\mathbf{T}(\mathcal{O}_0) = t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1$.

– Then by substitution we get

$$\begin{aligned}\mathbf{T}(P) &= x(t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1) + y(t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1) + t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1 \\ &= \left[t_{1,1}\vec{i}_1 + t_{2,1}\vec{j}_1 \;\; t_{1,2}\vec{i}_1 + t_{2,2}\vec{j}_1 \;\; t_{1,3}\vec{i}_1 + t_{2,3}\vec{j}_1 + \mathcal{O}_1\right]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \left[\vec{i}_1 \; \vec{j}_1 \; \mathcal{O}_1\right]\begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\end{aligned}$$

Using $\mathbf{M}_T$ to denote the matrix, we see that $F_0 = F_1\mathbf{M}_T$

- Let $\mathbf{T}(P) = P' = x'\vec{i}_1 + y'\vec{j}_1 + \mathcal{O}_1$
  In vector form this is

$$P' = \left[\vec{i}_1 \; \vec{j}_1 \; \mathcal{O}_1\right]\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$= \left[\vec{i}_1\ \vec{j}_1\ \mathcal{O}_1\right] \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

So we see that

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can write this in shorthand – $\mathbf{p}' = \mathbf{M}_T\mathbf{p}$

- $\mathbf{M}_T$is the *matrix representation* of **T**
  - The first column of $\mathbf{M}_T$represents $\mathbf{T}(\vec{i}_0)$
  - The second column of $\mathbf{M}_T$represents $\mathbf{T}(\vec{j}_0)$
  - The third column of $\mathbf{M}_T$represents $\mathbf{T}(\mathcal{O}_0)$

- *Translation*
  - Points are transformed as $[x'\ y'\ 1]^T = [x\ y\ 1]^T + [\Delta x\ \Delta y\ 0]^T$.
  - Vectors don't change.
  - Thus translation is affine but not linear.
    If it were linear, we would have $\mathbf{T}(P+Q) = \mathbf{T}(P) + \mathbf{T}(Q)$, but point addition is undefined.
  - Translation can be applied to sums of vectors and vector-point sums.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $T(\Delta x, \Delta y)$

- *Scale*
  - Linear transform — applies equally to points and vectors
  - Points transform as $[x'\ y'\ 1]^T = [xS_x\ yS_y\ 1]^T$.
  - Vectors transform as $[x'\ y'\ 0]^T = [xS_x\ yS_y\ 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} xS_x \\ yS_y \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $S(S_x, S_y)$
  - Note that this is *origin sensitive*.
  - How do you do reflections?

- *Rotate*
  - Linear transform — applies equally to points and vectors
  - Points transform as
    $[x'\ y'\ 1]^T = [x\cos(\theta) - y\sin(\theta)\ \ x\sin(\theta) + y\cos(\theta)\ \ 1]^T$.
  - Vectors transform as
    $[x'\ y'\ 0]^T = [x\cos(\theta) - y\sin(\theta)\ \ x\sin(\theta) + y\cos(\theta)\ \ 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $R(\theta)$
  - Note that this is *origin sensitive*.

- *Shear*
  - Linear transform — applies equally to points and vectors
  - Points transform as $[x'\ y'\ 1]^T = [x + \alpha y\ \ y + \beta x\ \ 1]^T$.
  - Vectors transform as $[x'\ y'\ 0]^T = [x + \alpha y\ \ y + \beta x\ \ 0]^T$.
  - Matrix formulation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y + \beta x \\ 0 \end{bmatrix}$$

  - Shorthand for the above matrix: $Sh(\alpha, \beta)$

- Composition of Transformations
  - Now we have some basic transformations, how do we create and represent arbitrary affine transformations?
  - We can derive an arbitrary affine transform as a sequence of basic transformations, then compose the transformations
  - Example — scaling about an arbitrary point $[x_c\ y_c\ 1]^T$
    1. Translate $[x_c\ y_c\ 1]^T$ to $[0\ 0\ 1]$ $(T(-x_c, -y_c)$
    2. Scale $\left[x'\ y'\ 1\right]^T = S(S_x, S_y)\,[x\ y\ 1]^T$
    3. Translate $[0\ 0\ 1]^T$ back to $[x_c\ y_c\ 1]$ $(T(x_c, y_c))$
  - The sequence of transformation steps is
    $T(-x_c, -y_c) \circ S(S_x, S_y) \circ T(x_c, y_c)$

– In matrix form this is

$$
\begin{aligned}
\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} S_x & 0 & x_c(1 - S_x) \\ 0 & S_y & y_c(1 - S_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\end{aligned}
$$

– Note that the matrices are arranged from *right to left* in the order of the steps.

– The order is important (why)?

- Three Dimensional Transformations
  - A point is $\mathbf{p} = [x\ y\ z\ 1]$, a vector $\vec{v} = [x\ y\ z\ 0]$
  - Translation:

$$T(\Delta x, \Delta y, \Delta z) = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Scale:

$$S(S_x, S_y, S_z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

– Rotation - clockwise wrt observer on positive axis looking at origin:

* Right handed

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Left handed

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation about an arbitrary axis
  - Another way to look at this is to treat a point $\mathbf{p}$ as a vector $\mathbf{p} - \mathbf{O}$, and break $\mathbf{p}$ into components along the axis of rotation and orthogonal to it, call these $\vec{p_a}$ and $\vec{p_o}$ respectively.
  - Assume that $\vec{a}$ is a unit vector along the axis of rotation. Define a third vector $\vec{p_1}$ orthogonal to both $\vec{p_a}$ and $\vec{p_o}$ and the same length as $\vec{p_o}$ by $\vec{p_1} = \vec{p_o} \times \vec{a}$.
  - Now we can write the rotation of $\vec{p}$ by θ about $\vec{a}$ as transformation $\vec{p'} = \vec{p_a} + \vec{p_o} \cos\theta + \vec{p_1} \sin\theta$
  - This form applies to any axis of rotation, not just the principal coordinate axes.
  - Since transforming a point and transforming a basis are equivalent, we can just rotate the basis vectors of our frame using this formula to get a matrix for rotation about an arbitrary axis.

$$R_a(\theta) = \begin{bmatrix} a_x^2 + (1 - a_x^2)\cos(\theta) & a_x a_y(1 - \cos(\theta)) - a_z \sin(\theta) & a_x a_z(1 - \cos(\theta)) - a_y \sin(\theta) & 0 \\ a_x a_y(1 - \cos(\theta)) + a_z \sin(\theta) & a_y^2 + (1 - a_y^2)\cos(\theta) & a_y a_z(1 - \cos(\theta)) - a_x \sin(\theta) & 0 \\ a_x a_z(1 - \cos(\theta)) - a_y \sin(\theta) & a_y a_y(1 - \cos(\theta)) + a_x \sin(\theta) & a_z^2 + (1 - a_z^2)\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Normal Vectors

- Normal vectors are used to indicate surface orientation
- The normal at a point is often computed as a cross product of two vectors tangent to the surface at that point
- For parametric surfaces, $F(u,v) = (x,y,z)$, a natural way to obtain the tangent vectors is to compute $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, so the normal $\vec{N}$ is $\vec{N} = \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}$
- For polygons, the two tangents can be obtained by vectors between vertices of the polygon.
- Normal vectors do not transform the same way as ordinary vectors (such as tangent vectors).
  - By definition, for tangent vector $\vec{t}$ and normal vector $\vec{n}$, $\vec{n} \cdot \vec{t} = \vec{n}^T \vec{t} = 0$
  - Let $\mathbf{M}$ be a transformation matrix, so that $\vec{t'} = \mathbf{M}\vec{t}$
  - There must be a transformation matrix $\mathbf{S}$ such that $\vec{n'} = \mathbf{S}\vec{n}$ and $\vec{n'} \cdot \vec{t'} = \vec{n'}^T \vec{t'} = 0$
  - So

$$(\mathbf{S}\vec{n})^T \mathbf{M}\vec{t} = 0$$
$$\vec{n}^T \mathbf{S}^T \mathbf{M}\vec{t} = 0$$
$$\vec{n}^T \mathbf{I}\vec{t} = 0$$
$$\vec{n}^T \vec{t} = 0$$

– Therefore

$$\mathbf{S}^T\mathbf{M} = \mathbf{I}$$
$$\mathbf{S}^T = \mathbf{M}^{-1}$$
$$\mathbf{S} = \mathbf{M}^{-1^T}$$

– Thus, given a transform applied to vectors, its inverse transpose must be applied to their tangents to preserve orthogonality