Orthogonal Functions and Fourier Series

Vector Spaces

- Set of vectors
- Closed under the following operations
  - Vector addition: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$
  - Scalar multiplication: $s\,\mathbf{v}_1 = \mathbf{v}_2$
  - Linear combinations: $\sum_{i=1}^{n} a_i \mathbf{v}_i = \mathbf{v}$
- Scalars come from some field **F**
  - e.g. real or complex numbers
- Linear independence
- Basis
- Dimension

Vector Space Axioms

- Vector addition is associative and commutative
- Vector addition has a (unique) identity element (the **0** vector)
- Each vector has an additive inverse
  - So we can define vector subtraction as adding an inverse
- Scalar multiplication has an identity element (1)
- Scalar multiplication distributes over vector addition and field addition
- Multiplications are compatible (a(b**v**)=(ab)**v**)

Coordinate Representation

- Pick a basis, order the vectors in it, then all vectors in the space can be represented as sequences of coordinates, i.e. coefficients of the basis vectors, in order.
- Example:
  - Cartesian 3-space
  - Basis: [**i** **j** **k**]
  - Linear combination: $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
  - Coordinate representation: $[x \quad y \quad z]$

$$a[x_1 \quad y_1 \quad z_1] + b[x_2 \quad y_2 \quad z_2] = [ax_1 + bx_2 \quad ay_1 + by_2 \quad az_1 + bz_2]$$

Functions as vectors

- Need a set of functions closed under linear combination, where
  - Function addition is defined
  - Scalar multiplication is defined
- Example:
  - Quadratic polynomials
  - Monomial (power) basis: $[\mathbf{x}^2 \quad \mathbf{x} \quad \mathbf{1}]$
  - Linear combination: $a\mathbf{x}^2 + b\mathbf{x} + c$
  - Coordinate representation: $[a \quad b \quad c]$

Metric spaces

- Define a *(distance) metric* $d(\mathbf{v}_1, \mathbf{v}_2) \Rightarrow R$ s.t.
  - d is nonnegative $\forall \mathbf{v}_i, \mathbf{v}_j \in \mathbf{V} : d(\mathbf{v}_i, \mathbf{v}_j) \geq 0$
  - d is symmetric $\forall \mathbf{v}_i, \mathbf{v}_j \in \mathbf{V} : d(\mathbf{v}_i, \mathbf{v}_j) = d(\mathbf{v}_j, \mathbf{v}_i)$
  - Indiscernibles are identical

$$\forall \mathbf{v}_i, \mathbf{v}_j \in \mathbf{V} : \; d(\mathbf{v}_i, \mathbf{v}_j) = 0 \;\Leftrightarrow\; \mathbf{v}_i = \mathbf{v}_j$$

  - The triangle inequality holds

$$\forall \mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k \in \mathbf{V} : d(\mathbf{v}_i, \mathbf{v}_j) + d(\mathbf{v}_j, \mathbf{v}_k) \geq d(\mathbf{v}_i, \mathbf{v}_k)$$

Normed spaces

- Define the *length* or *norm* of a vector $\|\mathbf{v}\|$
  - Nonnegative $\forall \mathbf{v} \in \mathbf{V} : \|\mathbf{v}\| \geq 0$
  - Positive definite $\|\mathbf{v}\| = 0 \Rightarrow \mathbf{v} = \mathbf{0}$
  - Symmetric $\forall \mathbf{v} \in \mathbf{V}, a \in F : \|a\, \mathbf{v}\| = |a|\, \|\mathbf{v}\|$
  - The triangle inequality holds

$$\forall \mathbf{v_i}, \mathbf{v_j} \in \mathbf{V} : \|\mathbf{v_i}\| + \|\mathbf{v_j}\| \geq \|\mathbf{v_i} + \mathbf{v_j}\|$$

- Banach spaces – normed spaces that are *complete* (no holes or missing points)
  - Real numbers form a Banach space, but not rational numbers
  - Euclidean *n*-space is Banach

Norms and metrics

- Examples of norms:
  - p norm:
    - p=1 manhattan norm
    - p=2 euclidean norm

$$\left(\sum_{i=1}^{D} \left| x_i \right|^p \right)^{\frac{1}{p}}$$

- Metric from norm

$$\mathrm{d}(\mathbf{v_1}, \mathbf{v_2}) = \left\| \mathbf{v_1} - \mathbf{v_2} \right\|$$

- Norm from metric if
  - d is homogeneous

$$\forall \mathbf{v_i}, \mathbf{v_j} \in \mathbf{V}, a \in F : \mathrm{d}(a\,\mathbf{v_i}, a\,\mathbf{v_j}) = |a|\,\mathrm{d}(\mathbf{v_i}, \mathbf{v_j})$$

  - d is translation invariant

$$\forall \mathbf{v_i}, \mathbf{v_j}, \mathbf{t} \in \mathbf{V} : \mathrm{d}(\mathbf{v_i}, \mathbf{v_j}) = \mathrm{d}(\mathbf{v_i} + \mathbf{t}, \mathbf{v_j} + \mathbf{t})$$

then

$$\left\| \mathbf{v} \right\| = \mathrm{d}(\mathbf{v}, \mathbf{0})$$

Inner product spaces

- Define *[inner, scalar, dot] product* $\left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle \Rightarrow R$ (for real spaces) s.t.

$$\left\langle \mathbf{v}_\mathrm{i} + \mathbf{v}_\mathrm{j}, \mathbf{v}_\mathrm{k} \right\rangle = \left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{k} \right\rangle + \left\langle \mathbf{v}_\mathrm{j}, \mathbf{v}_\mathrm{k} \right\rangle$$

$$\left\langle a\,\mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle = a\left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle$$

$$\left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle = \left\langle \mathbf{v}_\mathrm{j}, \mathbf{v}_\mathrm{i} \right\rangle$$

$$\left\langle \mathbf{v}, \mathbf{v} \right\rangle \geq 0$$

$$\left\langle \mathbf{v}, \mathbf{v} \right\rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$$

- For complex spaces: $\left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle = \overline{\left\langle \mathbf{v}_\mathrm{j}, \mathbf{v}_\mathrm{i} \right\rangle}$ $\quad \left\langle \mathbf{v}_\mathrm{i}, a\,\mathbf{v}_\mathrm{j} \right\rangle = \bar{a}\left\langle \mathbf{v}_\mathrm{i}, \mathbf{v}_\mathrm{j} \right\rangle$

- Induces a norm: $\left\| \mathbf{v} \right\| = \sqrt{\left\langle \mathbf{v}, \mathbf{v} \right\rangle}$

Some inner products

- Multiplication in R
- Dot product in Euclidean n-space

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \sum_{i=1}^{D} \mathbf{v}_{1,i} \mathbf{v}_{2,i}$$

- For real functions over domain [a,b]

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

- For complex functions over domain [a,b]

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

- Can add nonnegative weight function

$$\langle f, g \rangle_w = \int_a^b f(x) g(x) w(x) dx$$

Hilbert Space

- An inner product space that is complete wrt the induced norm is called a Hilbert space
- Infinite dimensional Euclidean space
- Inner product defines distances and angles
- Subset of Banach spaces

Orthogonality

- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are *orthogonal* if

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$$

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are *orthonormal* if they are orthogonal and $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 1$

- Orthonormal set of vectors

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{i,j}$$ (Kronecker delta)

Examples

- Linear polynomials over [-1,1] (orthogonal)
  - $B_0(x) = 1,\ \ B_1(x) = x$ $\qquad \displaystyle\int_{-1}^{1} x\,dx = 0$
  - Is x^2 orthogonal to these?
  - Is $\dfrac{3x^2+1}{2}$ orthogonal to them? (Legendre)

Fourier series

Cosine series

$$f(\theta) = \sum_{i=0}^{\infty} a_i C_i(\theta)$$

$$C_0(\theta) = 1, \quad C_1(\theta) = \cos(\theta), \quad C_n(\theta) = \cos(n\theta)$$

$$\langle C_m, C_n \rangle = \int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(\cos[(m+n)\theta] + \cos[(m-n)\theta])$$

$$= \left(\frac{1}{2(m+n)} \sin[(m+n)\theta] + \frac{1}{2(m-n)} \sin[(m-n)\theta] \right)\Bigg|_0^{2\pi} = 0$$

for $m \neq n \neq 0$

Fourier series

$$= \int_0^{2\pi} \left(\frac{1}{2}\cos(2n\theta) + \frac{1}{2} \right) d\theta = \left(\frac{1}{4n}\sin(2n\theta) + \frac{\theta}{2} \right) \Bigg|_0^{2\pi} = \pi \quad \text{for } m = n \neq 0$$

$$= \int_0^{2\pi} \frac{1}{2} 2\cos(0)\, d\theta = 2\pi \quad \text{for } m = n = 0$$

- Sine series $\qquad f(\theta) = \sum_{i=0}^{\infty} b_i S_i(\theta)$

$$S_0(\theta) = 0, \quad S_1(\theta) = \sin(\theta), \quad S_n(\theta) = \sin(n\theta)$$

$$\langle S_m, S_n \rangle = \int_0^{2\pi} \sin(m\theta)\sin(n\theta)\, d\theta = 0 \quad \text{for } m \neq n \text{ or } m = n = 0$$

$$= \pi \quad \text{for } m = n \neq 0$$

Fourier series

- Complete series $f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$

$$\langle C_m, S_n \rangle = \int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,d\theta = 0$$

- Basis functions are orthogonal but not orthonormal
- Can obtain a_n and b_n by projection

$$\langle f, C_k \rangle = \int_0^{2\pi} f(\theta)\cos(k\theta)\,d\theta = \int_0^{2\pi} \cos(k\theta)\,d\theta \sum_{n=0}^{\infty} a_i \cos(n\theta) + b_i \sin(n\theta)$$

$$= \int_0^{2\pi} a_k \cos^2(k\theta)\,d\theta = \pi\, a_k \quad (\text{or } 2\pi\, a_k \text{ for } k = 0)$$

Fourier series

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\,d\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

- Similarly for b_k

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)\,d\theta$$