Geometric optics

- Modern theories of light treat it as both a wave and a particle.
- We will take a combined and somewhat simpler view of light – the view of geometric optics.
- Here are the rules of geometric optics:
  - Light is a flow of photons with wavelengths. We'll call these flows “light rays.”
  - Light rays travel in straight lines in free space.
  - Light rays do not interfere with each other as they cross.
  - Light rays obey the laws of reflection and refraction.
  - Light rays travel from the light sources to the eye, but the physics is invariant under path reversal (reciprocity).
Synthetic pinhole camera

- The most common imaging model in graphics is the synthetic pinhole camera: light rays are collected through an infinitesimally small hole and recorded on an **image plane**.

- For convenience, the image plane is usually placed in front of the camera, giving a non-inverted 2D projection (image).
- Viewing rays emanate from the **center of projection** (COP) at the center of the lens (or pinhole).
- The image of an object point $P$ is at the intersection of the viewing ray through $P$ and the image plane.
Eye vs. light ray tracing

- Where does light begin?
- At the light: light ray tracing (a.k.a., forward ray tracing or photon tracing)

- At the eye: eye ray tracing (a.k.a., backward ray tracing)

- We will generally follow rays from the eye into the scene.
Precursors to ray tracing

- **Local illumination**
  - Cast one eye ray,
  - then shade according to light

- **Appel (1968)**
  - Cast one eye ray + one ray to light
The University of Texas at Austin CS384G – Computer Graphics  Don Fussell

Whitted ray-tracing algorithm

- In 1980, Turner Whitted introduced ray tracing to the graphics community.
  - Combines eye ray tracing + rays to light
  - Recursively traces rays

**Algorithm:**
1. For each pixel, trace a primary ray in direction $V$ to the first visible surface.
2. For each intersection, trace secondary rays:
   - Shadow rays in directions $L_i$ to light sources
   - Reflected ray in direction $R$.
   - Refracted ray or transmitted ray in direction $T$. 

The University of Texas at Austin CS384G – Computer Graphics  Don Fussell
Whitted algorithm (cont'd)

Let's look at this in stages:

Primary rays

Shadow rays

Reflection rays

Reflected rays
A ray is defined by an origin \( P \) and a unit direction \( d \) and is parameterized by \( t \):

\[
P + td
\]

Let \( I(P, d) \) be the intensity seen along that ray. Then:

\[
I(P, d) = I_{\text{direct}} + I_{\text{reflected}} + I_{\text{transmitted}}
\]

where

- \( I_{\text{direct}} \) is computed from the Phong model
- \( I_{\text{reflected}} = k_r I(Q, R) \)
- \( I_{\text{transmitted}} = k_t I(Q, T) \)

Typically, we set \( k_r = k_s \) and \( k_t = 1 - k_s \).
Reflection and transmission

- **Law of reflection:**
  \[ \theta_i = \theta_r \]

- **Snell's law of refraction:**
  \[ \eta_i \sin \theta_i = \eta_t \sin \theta_t \]

- where \( \eta_i \), \( \eta_t \) are **indices of refraction**.
The equation for the angle of refraction can be computed from Snell's law:

What happens when $\eta_i > \eta_t$?

When $\theta_t$ is exactly 90°, we say that $\theta_I$ has achieved the “critical angle” $\theta_c$.

For $\theta_I > \theta_c$, no rays are transmitted, and only reflection occurs, a phenomenon known as “total internal reflection” or TIR.
Reflected and transmitted rays

- For incoming ray $P(t) = P + td$
  - Compute input cosine and sine vectors $C_i$ and $S_i$
  - Reflected ray vector $R = C_i + S_i$
  - Compute output cosine and sine vectors $C_t$ and $S_t$
  - Transmitted ray vector $T = C_t + S_t$
Ray-tracing pseudocode

We build a ray traced image by casting rays through each of the pixels.

function traceImage (scene):
    for each pixel (i,j) in image
        \[ S = \text{pixelToWorld}(i,j) \]
        \[ P = \text{COP} \]
        \[ d = (S - P)/\|S - P\| \]
        \[ I(i,j) = \text{traceRay(scene, } P, d) \]
    end for
end function
function traceRay(scene, P, d):
    (t, N, mtrl) ← scene.intersect(P, d)
    Q ← ray(P, d) evaluated at t
    I = shade(q, N, mtrl, scene)
    R = reflectDirection(N, -d)
    I ← I + mtrl.kr * traceRay(scene, Q, R)
    if ray is entering object then
        n_i = index_of_air
        n_t = mtrl.index
    else
        n_i = mtrl.index
        n_t = index_of_air
        if (mtrl.k_t > 0 and notTIR(n_i, n_t, N, -d)) then
            T = refractDirection(n_i, n_t, N, -d)
            I ← I + mtrl.k_t * traceRay(scene, Q, T)
        end if
    end if
    return I
end function
Terminating recursion

- **Q**: How do you bottom out of recursive ray tracing?

- **Possibilities:**
Next, we need to calculate the color returned by the *shade* function.

```plaintext
function shade(mtrl, scene, Q, N, d):
    I ← mtrl.k_e + mtrl.k_a * scene->I_a
    for each light source L do:
        atten = L -> distanceAttenuation(Q)
        atten += L -> shadowAttenuation(scene, Q)
        I ← I + atten*(diffuse term + spec term)
    end for
    return I
end function
```
Shadow attenuation

- Computing a shadow can be as simple as checking to see if a ray makes it to the light source.
- For a point light source:

  function PointLight::shadowAttenuation(scene, P)
    d = (L.position - P).normalize()
    (t, N, mtrl) = scene.intersect(P, d)
    Q = ray.intersection(t)
    if Q is before the light source then:
      atten = 0
    else
      atten = 1
    end if
    return atten
  end function

- What if there are transparent objects along a path to the light source?
Ray-plane intersection

- We can write the equation of a plane as:
  \[ ax + by + cz + d = 0 \]

- The coefficients \( a, b, \) and \( c \) form a vector that is normal to the plane, \( \mathbf{n} = [a \ b \ c]^T \). Thus, we can re-write the plane equation as:
  \[ \mathbf{n} \cdot \mathbf{p}(t) + d = 0 \]
  \[ \mathbf{n} \cdot (\mathbf{P} + td) + d = 0 \]

- We can solve for the intersection parameter (and thus the point):
  \[ t = -\frac{\mathbf{n} \cdot \mathbf{P} + d}{\mathbf{n} \cdot \mathbf{d}} \]
Ray-triangle intersection

To intersect with a triangle, we first solve for the equation of its supporting plane:

\[ \mathbf{n} = (\mathbf{A} - \mathbf{C}) \times (\mathbf{B} - \mathbf{C}) \]
\[ d = -\left(\mathbf{n} \cdot \mathbf{A}\right) \]

Then, we need to decide if the point is inside or outside of the triangle.

- Solution 1: compute barycentric coordinates from 3D points.
- What do you do with the barycentric coordinates?
Barycentric coordinates

A set of points can be used to create an affine frame. Consider a triangle $ABC$ and a point $p$:

We can form a frame with an origin $C$ and the vectors from $C$ to the other vertices:

$$u = A - C \quad v = B - C \quad t = C$$

We can then write $P$ in this coordinate frame:

$$p = \alpha u + \beta v + t$$

The coordinates $(\alpha, \beta, \gamma)$ are called the **barycentric coordinates** of $p$ relative to $A$, $B$, and $C$. 
Computing barycentric coordinates

For the triangle example we can compute the barycentric coordinates of \( P \):

\[
\alpha A + \beta B + \gamma C = \begin{bmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}
\]

Cramer’s rule gives the solution:

\[
\alpha = \begin{vmatrix} p_x & B_x & C_x \\ p_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}, \quad \beta = \begin{vmatrix} A_x & p_x & C_x \\ A_y & p_y & C_y \\ 1 & 1 & 1 \end{vmatrix}, \quad \gamma = \begin{vmatrix} A_x & B_x & p_x \\ A_y & B_y & p_y \\ 1 & 1 & 1 \end{vmatrix}
\]

Computing the determinant of the denominator gives:

\[
B_x C_y - B_y C_x + A_y C_x - A_x C_y + A_x B_y - A_y B_x
\]
Cross products

Consider the cross-product of two vectors, \( \mathbf{u} \) and \( \mathbf{v} \). What is the geometric interpretation of this cross-product?

A cross-product can be computed as:

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_yv_z - u_zv_y)\mathbf{i} + (u_zv_x - u_xv_z)\mathbf{j} + (u_xv_y - u_yv_x)\mathbf{k}
\]

What happens when \( \mathbf{u} \) and \( \mathbf{v} \) lie in the \( x-y \) plane? What is the area of the triangle they span?
Barycentric coords from area ratios

Now, let’s rearrange the equation from two slides ago:

\[
B_x C_y - B_y C_x + A_y C_x - A_x C_y + A_x B_y - A_y B_x
\]

\[
= (B_x - A_x)(C_y - A_y) - (B_y - A_y)(C_x - A_x)
\]

The determinant is then just the \(z\)-component of

\((B-A) \times (C-A)\), which is two times the area of triangle \(ABC\)!

Thus, we find:

\[
\alpha = \frac{\text{SArea}(pBC)}{\text{SArea}(ABC)} \quad \beta = \frac{\text{SArea}(ApC)}{\text{SArea}(ABC)} \quad \gamma = \frac{\text{SArea}(ABp)}{\text{SArea}(ABC)}
\]

Where SArea(RST) is the signed area of a triangle, which can be computed with cross-products.
Ray-triangle intersection

- Solution 2: project down a dimension and compute barycentric coordinates from 2D points.

- Why is solution 2 possible? Why is it legal? Why is it desirable? Which axis should you “project away”?

The University of Texas at Austin CS384G – Computer Graphics  Don Fussell
Interpolating vertex properties

- The barycentric coordinates can also be used to interpolate vertex properties such as:
  - material properties
  - texture coordinates
  - normals

- For example:

\[ k_d(Q) = \alpha k_d(A) + \beta k_d(B) + \gamma k_d(C) \]

- Interpolating normals, known as Phong interpolation, gives triangle meshes a smooth shading appearance. (Note: don’t forget to normalize interpolated normals.)
Epsilons

- Due to finite precision arithmetic, we do not always get the exact intersection at a surface.
- **Q:** What kinds of problems might this cause?

- **Q:** How might we resolve this?
Intersecting with xformed geometry

- In general, objects will be placed using transformations. What if the object being intersected were transformed by a matrix $M$?
- Apply $M^{-1}$ to the ray first and intersect in object (local) coordinates!
Intersecting with xformed geometry

- The intersected normal is in object (local) coordinates. How do we transform it to world coordinates?