Barycentric Coordinates and Parameterization
Center of Mass

“Geometric center” of object
Center of Mass

“Geometric center” of object
Object can be balanced on CoM

How to calculate?
Finding the Center of Mass

Plumb line method
Special Case: Points

CoM is average

\[ c = \frac{1}{n} \sum_{i=1}^{n} p_i \]
Special Case: Points

CoM is average

\[ c = \frac{1}{n} \sum_{i=1}^{n} p_i \]

Center of mass is inside \textbf{convex hull}
Special Case: Points

CoM is average

\[ c = \frac{1}{n} \sum_{i=1}^{n} p_i \]

Center of mass is inside convex hull

What if points have different mass?
Special Case: Points
Special Case: Points

Weighted average

\[ c = \frac{\sum_{i=1}^{n} m_i p_i}{\sum_{i=1}^{n} m_i} \]

Still in convex hull

Scaling the masses doesn’t affect CoM
Special Case: Points

Weighted average

\[ c = \frac{\sum_{i=1}^{n} m_i p_i}{\sum_{i=1}^{n} m_i} \]

Still in convex hull

Scaling the masses doesn’t affect CoM

- can assume masses sum to one
Inverse Problem

Given three points $p_i$ and a target point $c$:

For what masses is $c$ the CoM?
Inverse Problem

Given three points \( p_i \) and a target point \( c \):
For what masses is \( c \) the CoM?

Special case 1: \( c = p_i \)
Inverse Problem

Given three points \( p_i \) and a target point \( c \):

For what masses is \( c \) the CoM?

Special case 1: \( c = p_i \)

\[
m_j = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}
\]
Inverse Problem

Given three points $p_i$ and a target point $c$:
For what masses is $c$ the CoM?

Special case 1: $c = p_i$

Special case 2: $c$ outside triangle
Inverse Problem

Given three points \( p_i \) and a target point \( c \):

For what masses is \( c \) the CoM?

Special case 1: \( c = p_i \)

Special case 2: \( c \) outside triangle

• not possible (needs antigravity…)
Inverse Problem

Given three points $p_i$ and a target point $c$:

For what masses is $c$ the CoM?

$$c = \sum_i m_i p_i, \quad \sum m_i = 1$$

Observation: $m_1 = 1 - m_2 - m_3$
Inverse Problem

Given three points \( p_i \) and a target point \( c \):

For what masses is \( c \) the CoM?

\[
c = (1 - m_2 - m_3)p_1 + m_2p_2 + m_3p_3
\]
Inverse Problem

Given three points $p_i$ and a target point $c$:

For what masses is $c$ the CoM?

\[ c = (1 - m_2 - m_3)p_1 + m_2 p_2 + m_3 p_3 \]
\[ c - p_1 = m_2 (p_2 - p_1) + m_3 (p_3 - p_1) \]
Inverse Problem

Given three points \( p_i \) and a target point \( c \):
For what masses is \( c \) the CoM?

\[
c = (1 - m_2 - m_3)p_1 + m_2 p_2 + m_3 p_3
\]

\[
c - p_1 = m_2 (p_2 - p_1) + m_3 (p_3 - p_1)
\]

\[
c - p_1 = \begin{bmatrix} p_2 - p_1 & p_3 - p_1 \end{bmatrix}_{2 \times 2} \begin{bmatrix} m_2 \\ m_3 \end{bmatrix}
\]
Inverse Problem

Given three points $p_i$ and a target point $c$:

For what masses is $c$ the CoM?

$$
\begin{bmatrix}
m_2 \\
m_3
\end{bmatrix} = \begin{bmatrix}
p_2 - p_1 & p_3 - p_1
\end{bmatrix}^{-1} \begin{bmatrix}
c - p_1
\end{bmatrix}
$$

These are barycentric coordinates of $c$.
Barycentric Coordinates

Can be interpreted as

- weighted point sum
  \[(1 - m_2 - m_3)p_1 + m_2p_2 + m_3p_3\]

- point in edge coordinates
  \[p_1 + m_2(p_2 - p_1) + m_3(p_3 - p_1)\]
Barycentric Coordinates

Properties:

\begin{itemize}
  \item $0 \leq m_i \leq 1$
  \item $0 \leq m_2 + m_3 \leq 1$
\end{itemize}
Barycentric Coordinates

Properties:

• $0 \leq m_i \leq 1$
• $0 \leq m_2 + m_3 \leq 1$
• corners are (0,0), (1,0), (0,1)
• unique for any inside point
Barycentric Coordinates

Properties:

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Barycentric Coordinates

Properties:
• $0 \leq m_i \leq 1$
• $0 \leq m_2 + m_3 \leq 1$
• corners are (0,0), (1,0), (0,1)
• unique for any inside point

Why do we care?
Barycentric Interpolation

Extends any function from corners to triangle

\[ f_c = (1 - m_2 - m_3)f_1 + m_2 f_2 + m_3 f_3 \]
Barycentric Interpolation

Extends any function from corners to triangle

- colors
- normals
- whatever

\[ f_c = (1 - m_2 - m_3) f_1 + m_2 f_2 + m_3 f_3 \]
Negative Barycentric Coordinates

Points outside triangle also have coords

$p_1$ (0.5, -0.5)
Negative Barycentric Coordinates

Points outside triangle also have coords

Alternate inside-triangle check:
- compute barycentric coords
- check they’re valid
Barycentric Coords in 3D

Given $c$ in plane of tri: find coords with

$$c - p_1 = m_2(p_2 - p_1) + m_3(p_3 - p_1)$$
Barycentric Coords in 3D

Given \( c \) in plane of tri:
find coords with

\[
c - p_1 = m_2(p_2 - p_1) + m_3(p_3 - p_1)
\]

Problem: too many equations!!
Barycentric Coords in 3D

Given $c$ in plane of tri: find coords with

$$c - p_1 = m_2(p_2 - p_1) + m_3(p_3 - p_1)$$

Problem: too many equations!!
Can we eliminate one of the variables?
Barycentric Coords in 3D

Given \( c \) in plane of tri:
find coords with

\[
\begin{align*}
  c - p_1 &= m_2(p_2 - p_1) + m_3(p_3 - p_1) \\
  (p_3 - p_1) \times (c - p_1) &= m_2(p_3 - p_1) \times (p_2 - p_1)
\end{align*}
\]

Both sides vectors in normal direction
Barycentric Coords in 3D

Given \( c \) in plane of tri:
find coords with

\[
\begin{align*}
  c - p_1 &= m_2 (p_2 - p_1) + m_3 (p_3 - p_1) \\
  (p_3 - p_1) \times (c - p_1) &= m_2 (p_3 - p_1) \times (p_2 - p_1)
\end{align*}
\]

Both sides vectors in normal direction

\[
[(p_3 - p_1) \times (c - p_1)] \cdot \hat{n} = m_2 [(p_3 - p_1) \times (p_2 - p_1)] \cdot \hat{n}
\]
Barycentric Coords in 3D

Given \( c \text{ in plane of tri:} \)
find coords with

\[
c - p_1 = m_2(p_2 - p_1) + m_3(p_3 - p_1)
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(p_3 - p_1) \times (c - p_1) = m_2(p_3 - p_1) \times (p_2 - p_1)
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Both sides vectors in normal direction

\[
[(p_3 - p_1) \times (c - p_1)] \cdot \hat{n} = m_2 [(p_3 - p_1) \times (p_2 - p_1)] \cdot \hat{n}
\]

\[
[(p_2 - p_1) \times (c - p_1)] \cdot \hat{n} = m_3 [(p_2 - p_1) \times (p_3 - p_1)] \cdot \hat{n}
\]
Barycentric Coords in 3D

Given \( c \) in plane of tri: find coords with

\[
m_2 = \frac{[(p_3 - p_1) \times (c - p_1)] \cdot \hat{n}}{[(p_3 - p_1) \times (p_2 - p_1)] \cdot \hat{n}}
\]

\[
m_3 = \frac{[(p_2 - p_1) \times (c - p_1)] \cdot \hat{n}}{[(p_2 - p_1) \times (p_3 - p_1)] \cdot \hat{n}}
\]
Barycentric Coords in 3D

Given \( c \) in plane of tri: find coords with

\[
m_2 = \frac{\left((p_3 - p_1) \times (c - p_1)\right) \cdot \hat{n}}{\left((p_3 - p_1) \times (p_2 - p_1)\right) \cdot \hat{n}}
\]

\[
m_3 = \frac{\left((p_2 - p_1) \times (c - p_1)\right) \cdot \hat{n}}{\left((p_2 - p_1) \times (p_3 - p_1)\right) \cdot \hat{n}}
\]

What if \( c \) is not in the plane of triangle?
Ray Tracing Triangles

1. Find point where ray hits triangle plane
2. Calculate barycentric coordinates
3. Check coords valid
4. Linearly interpolate normals etc.
5. Shade pixel
Beyond Triangles

Much carries over…

\[ c = (1 - m_2 - m_3 - m_4)p_1 + m_2p_2 + m_3p_3 + m_4p_4 \]

Are coords still unique?
Beyond Triangles

Much carries over…

\[ m_4 = 0 \]

\[ c = (1 - m_2 - m_3 - m_4)p_1 + m_2p_2 + m_3p_3 + m_4p_4 \]

Are coords still unique? No!
Beyond Triangles

Much carries over…

\[ m_4 = 0 \]

\[ c = (1 - m_2 - m_3 - m_4) p_1 + m_2 p_2 + m_3 p_3 + m_4 p_4 \]

Are coords still unique? No!

Many generalized barycentric coords schemes exist
Barycentric Coords as a Map

Maps from a triangle in 2D to 3D triangle

Called *parameterization* of triangle
Barycentric Coords as a Map

Maps from a triangle in 2D to 3D triangle

Called **parameterization** of triangle

- from now on, 2D coords are $u$ and $v$
Parameterization

Map between **region of plane** and **arbitrary surface**

why do we want to do this?
Parameterization

Map between region of plane and arbitrary surface

Can then use parameterization to paint image on 3D surface: texture map
Texture Map

Parameterization == texture map
== UV coordinates
== UV unwrapping
Texture Map

Parameterization == texture map
== UV coordinates
== UV unwrapping

Usually means assigning U and V coordinates to every pixel
Texture Map

Parameterization == texture map
    == UV coordinates
    == UV unwrapping

Usually means assigning U and V coordinates to every pixel
Or U and V for every vertex, then interpolate
Parameterization History

How to parameterize the earth (sphere)?

Very practical, important problem in Middle Ages…
Latitude & Longitude

Distorts areas and angles
Planar Projection

Covers only half of the earth
Distorts areas and angles
Stereographic Projection

Distorts areas
Albers Projection

Preserves areas, distorts aspect ratio
Fuller Parameterization
No Free Lunch

Every parameterization of the earth either:
• distorts areas
• distorts distances
• distorts angles
Good Parameterizations

- low area distortion
- low angle distortion
- no obvious seams
- one piece
Soup Parameterization
Planar Parameterization

Project surface onto plane
Planar Parameterization

Project surface onto plane
• quite useful in practice
Planar Parameterization

Project surface onto plane
• quite useful in practice
• only partial coverage
• bad distortion when normals perpendicular
Planar Parameterization

In practice: combine multiple views
Cube Map
Cylindrical Parameterization
Conformal Parameterization

Conformal = angle-preserving
Conformal Parameterization

Conformal = angle-preserving

Riemann mapping theorem

• can map any surface conformally
Conformal Parameterization

Conformal = angle-preserving

Riemann mapping theorem
• can map any surface conformally

Area distortion can be bad
Texture Atlas

Break up surface into easy pieces, parameterize separately
Texture Atlas

Some automatic methods exist...

but often artists hand-paint UV coords
Projection Mapping
Projection Mapping

Scan 3D geometry, compute texture map

Then, project anything you want on object