Projective geometry- 2D

Acknowledgements

Marc Pollefeys: for allowing the use of his excellent slides on this topic

http://www.cs.unc.edu/~marc/mvg/

Richard Hartley and Andrew Zisserman, "Multiple View Geometry in Computer Vision"

Homogeneous coordinates

Homogeneous representation of lines

$ax + by + c = 0 \qquad (a,b,c)^\mathsf{T}$

$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^\mathsf{T} \sim k(a,b,c)^\mathsf{T}$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3 - (0,0,0)^\mathsf{T}$ forms $\mathbf{P}^2$

Homogeneous representation of points

$\mathrm{x} = (x, y)^\mathsf{T}$ on $\mathrm{l} = (a,b,c)^\mathsf{T}$ if and only if $ax + by + c = 0$

$(x,y,1)(a,b,c)^\mathsf{T} = (x,y,1)\mathrm{l} = 0 \qquad (x,y,1)^\mathsf{T} \sim k(x,y,1)^\mathsf{T}, \forall k \neq 0$

The point x lies on the line l if and only if $\mathrm{x}^\mathsf{T}\mathrm{l} = \mathrm{l}^\mathsf{T}\mathrm{x} = 0$

Homogeneous coordinates $(x_1, x_2, x_3)^\mathsf{T}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^\mathsf{T}$

Points from lines and vice-versa

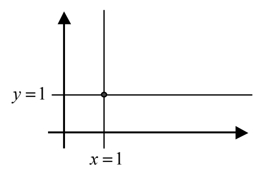
Intersections of lines

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Line joining two points

The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example

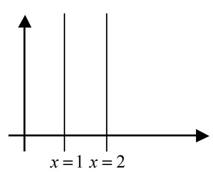


Ideal points and the line at infinity

Intersections of parallel lines

$\mathbf{l} = (a,b,c)^\mathsf{T}$ and $\mathbf{l}' = (a,b,c')^\mathsf{T}$ $\quad \mathbf{l} \times \mathbf{l}' = (b,-a,0)^\mathsf{T}$

Example



Ideal points $(x_1, x_2, 0)^\mathsf{T}$ Note that this set lies on a single line,

Line at infinity $\mathbf{l}_\infty = (0,0,1)^\mathsf{T}$

$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$ Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

Summary

The set of ideal points lies on the *line at infinity*, $\mathbf{l}_\infty = (0,0,1)^T$

$\mathbf{l} = (a,b,c)^T$ intersects the line at infinity in the ideal point $(b,-a,0)^T$

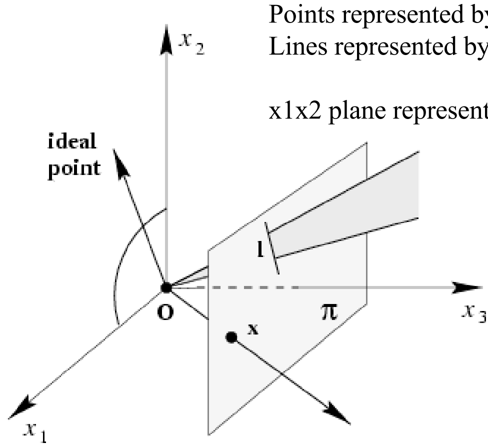
A line $\mathbf{l}' = (a,b,c')^T$ parallel to $\mathbf{l}$ also intersects $\mathbf{l}_\infty$ in the same ideal point, irrespective of the value of c'.

In inhomogeneous notation, $(b,-a)^T$ is a vector tangent to the line.
It is orthogonal to (a, b) -- the line normal.
Thus it represents the line direction.
As the line's direction varies, the ideal point $(b,-a)^T$ varies over $\mathbf{l}_\infty$.
--> line at infinity can be thought of as the set of directions of lines in the plane.

A model for the projective plane

Points represented by rays through origin
Lines represented by planes through origin

x1x2 plane represents line at infinity



exactly one line through two points
exaclty one point at intersection of two lines

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^T\mathbf{l}=0 \longleftrightarrow \mathbf{l}^T\mathbf{x}=0$$

$$\mathbf{x}=\mathbf{l}\times\mathbf{l}' \longleftrightarrow \mathbf{l}=\mathbf{x}\times\mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2+bxy+cy^2+dx+ey+f=0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

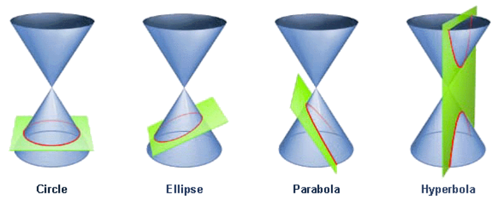
$$ax_1^2+bx_1x_2+cx_2^2+dx_1x_3+ex_2x_3+fx_3^2=0$$

or in matrix form

$$\boxed{\mathbf{x}^T\mathbf{C}\,\mathbf{x}=0} \text{ with } \mathbf{C}=\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a:b:c:d:e:f\}$

Conics …



http://ccins.camosun.bc.ca/~jbritton/jbconics.htm

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

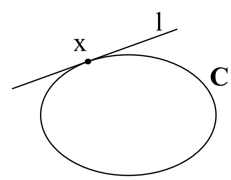
$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^\mathsf{T}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to **C** at point x on **C** is given by l=**C**x



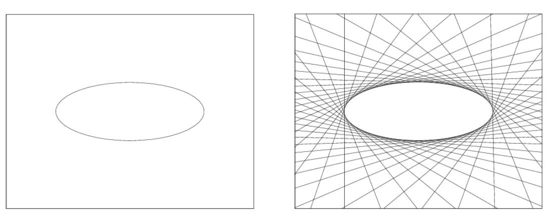
Dual conics

A line tangent to the conic **C** satisfies $l^T C^* l = 0$

In general (**C** full rank): $C^* = C^{-1}$

C* : Adjoint matrix of C.

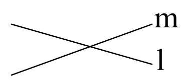
Dual conics = line conics = conic envelopes



Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{l}\mathrm{m}^{\mathsf{T}} + \mathrm{m}\mathrm{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{l}\mathrm{l}^{\mathsf{T}}$$

Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Projective transformations

Definition:

A *projectivity* is an invertible mapping h from P² to itself such that three points $\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3$ lie on the same line if and only if $h(\mathrm{x}_1),h(\mathrm{x}_2),h(\mathrm{x}_3)$ do.

Theorem:

A mapping $h:\mathrm{P}^2\rightarrow\mathrm{P}^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(\mathrm{x})=\mathbf{H}\mathrm{x}$

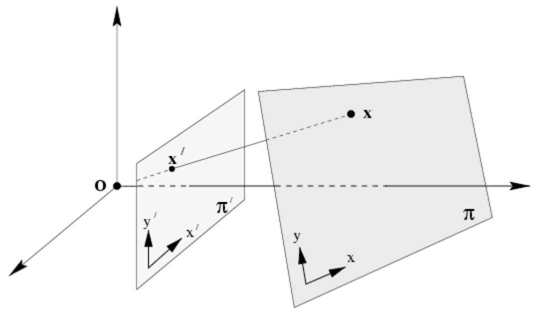
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

8DOF

projectivity=collineation=projective transformation=homography

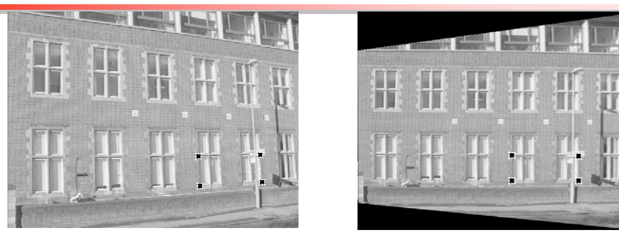
Mapping between planes



central projection may be expressed by x'=Hx

(application of theorem)

Removing projective distortion



select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

Transformation of lines and conics

For a point transformation

$$x' = \mathbf{H}\, x$$

Transformation for lines

$$l' = \mathbf{H}^{-T}\, l$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T}$$

Distortions under center projection

Similarity: squares imaged as squares.
Affine: parallel lines remain parallel; circles become ellipses.
Projective: Parallel lines converge.

Class I: Isometries

(*iso*=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

(*isometry + scale*)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

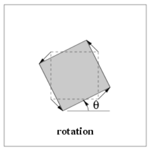
4DOF (1 scale, 1 rotation, 2 translation)

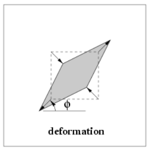
also know as *equi-form* (shape preserving)
metric structure = structure up to similarity (in literature)

Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$





$$x' = \mathbf{H}_A x = \begin{bmatrix} \mathbf{A} & t \\ 0^T & 1 \end{bmatrix} x$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Class VI: Projective transformations

$$x' = \mathbf{H}_P x = \begin{bmatrix} \mathbf{A} & t \\ v^T & v \end{bmatrix} x \qquad v = (v_1, v_2)^T$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line (ratio of ratio)

Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon.

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

decomposition unique (if chosen s>0) $\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

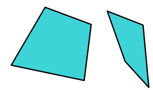
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Overview transformations

Projective
8dof

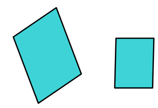
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

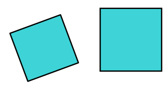
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

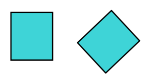
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.