A Theory of Abstraction for Arrays

Steven German

IBM T.J. Watson Research Center

October 2011

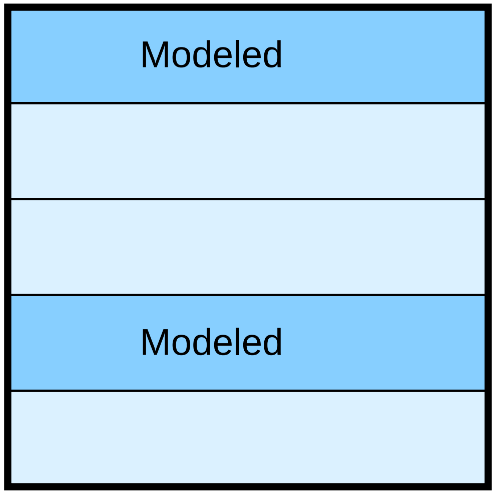
The Problem of Verifying Systems with Arrays

- Large arrays are often a barrier to verifying hardware designs
- Many previous approaches to abstracting arrays
- Abstracting arrays over a bounded time interval
  - Many approaches, including: Velev *et al* 1977; Ganai *et al* 2004 and 2005; Manolios *et al* 2006
- Prefer methods that:
  - Build unbounded-time sequential models
  - Are fully automatic
- Most directly related previous approach by Bjesse [FMCAD 2008]
- Limitations of previous approach
  - No reduction when latency from array read to output is unbounded
  - Clock gating introduces unbounded latency

New Results of This Paper

- New mathematical principle for abstraction of arrays
  - New principle allows unbounded latency from array read to output
  - Based on Small Model Theorem for a word-level logic with arrays
  - Previous approaches are based on principle of overapproximating behavior
- Automatic algorithm for constructing abstract models
  - Algorithm can build small abstract models for complex industrial designs
- Abstract models are sound and complete for safety properties
- To obtain these results, need to develop mathematical theory
- Details are in a longer version of paper, available from author

Traditional Abstract Models of Arrays



Modeled address: Normal array semantics

Unmodeled address: Nondeterministic value

1. Replace array with smaller array that overapproximates
   - Sound for safety properties
2. Restrict safety property to cases where modeled addresses are read

$$p \quad \Longrightarrow \quad modeled \rightarrow p$$

Unbounded Latency

- Bjesse 2008 shows how to define $modeled(k)$ to mean

  "k cycles in past, a modeled address was read"

  – Example: $modeled(2) \wedge modeled(3) \rightarrow p$

  – Solution for bounded latency

- For unbounded latency, not helpful to use

  "Array reads at all times in past were to modeled addresses"

  – Only true in unabstracted model

- New idea: Define a formula that means

  "Output at current time does not **depend** on reading unmodeled array addresses at any time in past"

A New Approach to Array Abstraction

- Read, write to modeled addresses have normal semantics
- Choose modeled addresses nondeterministically (as in Bjesse 2008)
- Read to unmodeled addresses returns special value $\perp$
- Value $\perp$ propagates according to semantic rules
- Property p ➡ $p \neq \perp \rightarrow p = \text{true}$
- Sound provided:

  At all times, For all inputs,

  Number of array addresses p depends on $\leq$ Number of modeled addresses

- If there is a counterexample to safety property p, some nondeterministic choice of modeled addresses finds the counterexample
- Goal of talk is to make these ideas more clear

Steps to Realize New Approach

1. Define mathematical meaning of dependence of a signal on an array address
2. Give automatic method for determining that at all times, for all inputs,

$$\text{signal } p \text{ depends on } \leq n \text{ array addresses}$$

3. Show that the proof method is sound

- Mathematics is different from traditional approach, where soundness follows easily from overapproximate behavior on ummodeled addresses

A Term Logic with Arrays

Two kinds of expressions: *signal expressions* and *array expressions*.

- Signal expressions
  1. Signal variable
     - Represents word level signal
  2. $op(e_1, \ldots, e_k)$, where $e_1, \ldots, e_k$ are signal expressions
     - Represents block of combinational logic
  3. $mux(control,\ data_1,\ data_2)$, where $control,\ data_1,\ data_2$ are signal expressions. Use data forwarding properties in abstract models.
  4. $a[addr]$, where a is an array expression and $addr$ is a signal expression.
- Array expressions
  1. Array variable
  2. $write(a,\ addr,\ value)$, where a is an array expression and $addr,\ value$ are signal expressions

Signal and Array Values

- Finite set of signal values (word-level), V
- Bottom value, $\perp \notin V$, represents subscripting array out of range
- Extended set of signal values, $V^{+} = V \cup \{\perp\}$
- Set of array values, $V \to V^{+}$

States

A state σ is a function mapping all signal and array variables to values.

- For signal variable s, $\sigma(s) \in V$
- For array variable a, $\sigma(a) \in (V \to V)$
- States are used to represent initial conditions of systems

Semantics of Expressions

The semantics of expressions maps a state and an expression to a value.

- For signal expression se, $\sigma[\![se]\!] \in V^+$
- For array expression ae, $\sigma[\![ae]\!] \in (V \rightarrow V^+)$
- Purpose of semantics is to allow reasoning about system with reduced arrays
- Reading an array outside its domain produces bottom value $\perp$
- Writing an array to an address in V outside domain of array, does not change value of array
- Writing an array with address $\perp$ causes all elements of array to be $\perp$
- Operator expression $op(e_1, \ldots, e_n)$ produces output $\perp$ if any input is $\perp$
- Multiplexor $mux(e_1,\ e_2,\ e_3)$ produces output $\perp$ if control input e_1 is $\perp$ or selected input $e_2,\ e_3$ is $\perp$

Operational Semantics

- A system $\mathcal{M}$ is defined by state variables and next-state expressions

  $\mathcal{N}(s)$ is the next-state expression for state variable s

- Define s^k to be an expression for state variable s at time k

  $s^0 = s$

  s^k is k^{th} expansion of $\mathcal{N}(s)$

- Value of s at time k in initial state σ is $\sigma[\![s^k]\!]$

Checking Safety Properties

- System $\mathcal{M}$
- Safety property represented by output signal p ($p = 1$ iff property is true)
- Let $\mathcal{T}$ be a set of states
- Safety property p holds over all initial states in $\mathcal{T}$ iff

$$\forall \sigma \in \mathcal{T},\ \forall k \geq 0:\ \sigma[\![p^k]\!] = 1$$

- This check corresponds to model checking the design on arrays of original size
  - Construct circuit representation of $\sigma[\![p^k]\!]$ using the next-state expressions
- We will show how to check safety properties over arrays of a smaller size

Essential Array Indices

Depending on the state, some indices of an array do not need to be evaluated

- Example: Let E be the expression $write(write(a, e1, a[1]), e2, a[2])\ [f]$

  If $\sigma[\![f]\!] = \sigma[\![e2]\!] \implies \{f, 2\}$

  If $\sigma[\![f]\!] \neq \sigma[\![e2]\!] \wedge \sigma[\![f]\!] = \sigma[\![e1]\!] \implies \{f, 1\}$

  If $\sigma[\![f]\!] \neq \sigma[\![e2]\!] \wedge \sigma[\![f]\!] \neq \sigma[\![e1]\!] \implies \{f\}$

  In every state, set of needed index expressions is an element of the set

  $S = \{\{f\}, \{f, 1\}, \{f, 2\}\}$

For general case, we can define a function

- Essential Indices, eindx$(exp,\ \sigma,\ array_variable) \mapsto \{array_indices\} \subseteq V$
  - Array indices that must be read from $array\_variable$ to evaluate exp in σ

- Idea of Small Model Theorem

  For any state σ, no matter how large the array a in σ, there exists a state σ' where a has size 2, and $\sigma'[\![E]\!] = \sigma[\![E]\!]$

Small Model Using Essential Indices

The semantics $\sigma[\![exp]\!]$ and the function eindx$(exp,\ \sigma,\ a)$ have the following relationship:

Lemma. For all exp, σ, a, there exists a state σ' such that

- $\sigma' \leq \sigma$
- For all array variables a, $\mathrm{dom}(\sigma'(a)) =$ eindx$(exp,\ \sigma,\ a)$
- $\sigma'[\![exp]\!] = \sigma[\![exp]\!]$

- The state σ' is a small model for the value of expression exp in state σ

Definition. A state σ' is called a *substate* of σ, written $\sigma' \leq \sigma$ iff

- For all signal variables s, $\sigma'(s) = \sigma(s)$, and
- For all array variables a, $\sigma'(a) \subseteq \sigma(a)$

Checking Safety Properties with Small Arrays

- Let $\mathcal{T}$ be a set of states and a an array variable such that a has size n for all states in $\mathcal{T}$
- Let m be

$$m \;=\; \max_{\sigma\in\mathcal{T}} \; \max_{k\geq 0} \; |\mathsf{eindx}(p^k, \sigma, a)| \;\leq\; n$$

$\forall\sigma\in\mathcal{T}$, $\forall k\geq 0$, there is a state σ' where a has size m and $\sigma'[\![p^k]\!] = \sigma[\![p^k]\!]$

- Let $\mathcal{T}'$ be the set of substates of states in $\mathcal{T}$ where a has size m
- Assume for all initial states in $\mathcal{T}$, that p is evaluated without subscript errors
- Then, $(p = 1)$ is always true in executions from initial states in $\mathcal{T}$

  iff $(p = 1 \vee p = \bot)$ is always true in executions from initial states in $\mathcal{T}'$
- Model where array a has size m is sound and complete for safety property p
- See conference paper for proof

Size of the Abstract Model

- The function $\max_{k \geq 0} \max_{\sigma} |\mathsf{eindx}(p^k, \sigma, a)|$ is difficult to compute!
- Case splitting overapproximates $\max_{\sigma} |\mathsf{eindx}(p^k, \sigma, a)|$, for a fixed k
- Example: Let E be the expression $\mathit{write}(\mathit{write}(a,\ e1,\ a[1]),\ e2,\ a[2])\ [f]$

  If $\sigma[\![f]\!] = \sigma[\![e2]\!] \implies \{f,\ 2\}$

  If $\sigma[\![f]\!] \neq \sigma[\![e2]\!] \wedge \sigma[\![f]\!] = \sigma[\![e1]\!] \implies \{f,\ 1\}$

  If $\sigma[\![f]\!] \neq \sigma[\![e2]\!] \wedge \sigma[\![f]\!] \neq \sigma[\![e1]\!] \implies \{f\}$

  In every state, set of index expressions is an element of the two-level set

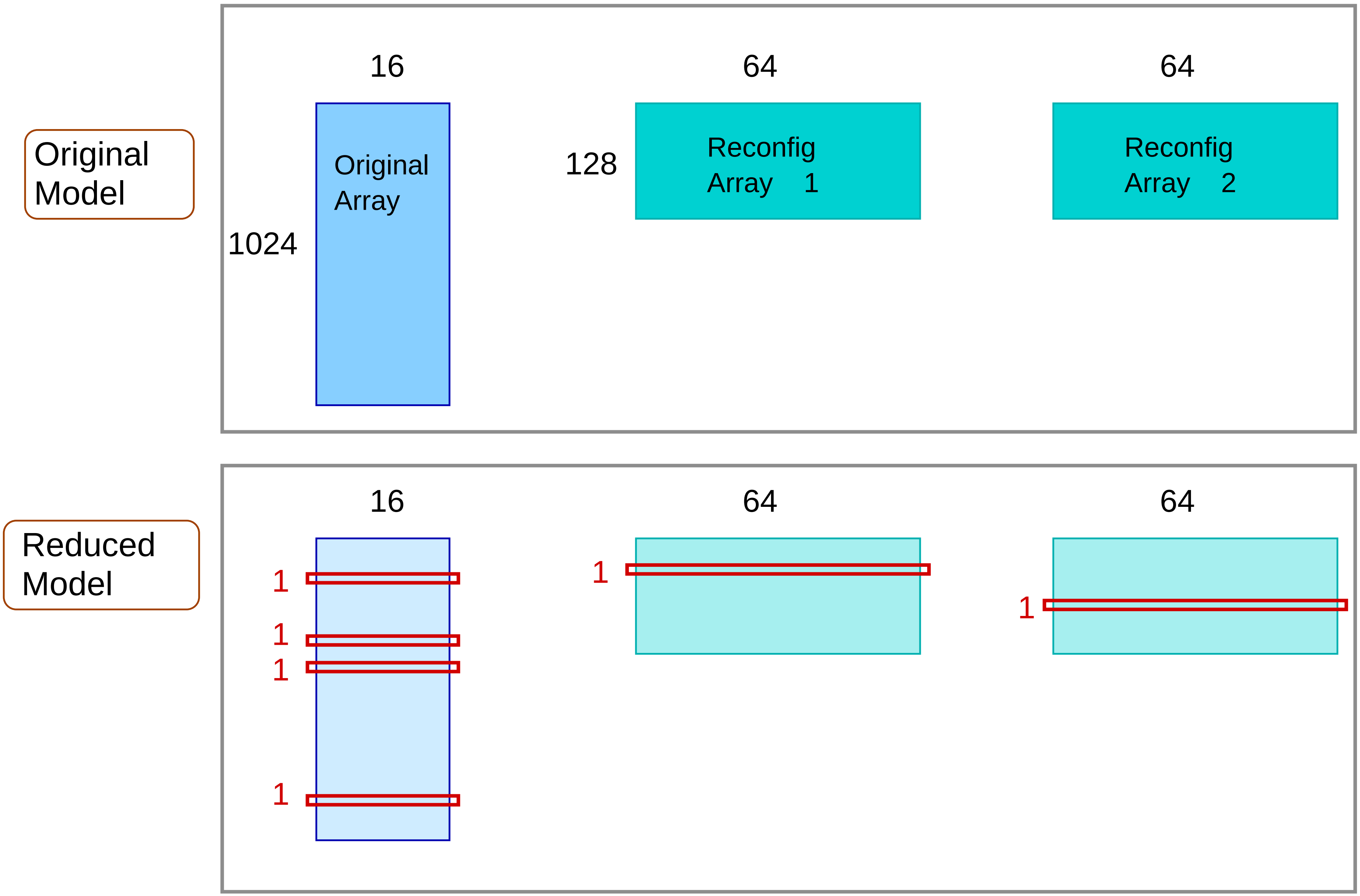
  $S = \{\{f\},\ \{f,\ 1\},\ \{f,\ 2\}\}$
- The set S overapproximates eindx $\quad \forall \sigma\ \exists s \in S : \mathsf{eindx}(E,\ \sigma,\ a) \subseteq \sigma(s)$
- Recursive algorithm constructs the two-level set for any expression
- A fixed point computation can find a set of expressions that overapproximates the largest set of index expressions over the sequence $p^0,\ p^1,\ p^2, \ldots$

Industrial Examples

- Implementation is in development
- Preliminary results with algorithm show reduction in cases that could not be reduced by previous methods
- Set of 255 examples not solvable in 24 hours by other methods
  - Reduced some arrays in 85 examples (33%)
  - Completely solved 33 examples in ≤ 2 hours

Sequential Equivalence of Systems with Arrays

- Due to physical limits, designers may split large array into smaller arrays
- In simple cases, new design has arrays with same number of rows, fewer columns
- Harder case is when new design has array with different number of rows



Original Model: 32912 registers

Reduced Model: 401 registers

Summary

- New theory of array abstraction based on Small Model Theorem
- Reduced size of arrays is computed automatically by static analysis
- Early experimental results are encouraging
- Planned Improvements
  - Improve the accuracy of the array size estimate
- Longer version of paper is available

Extra Slides

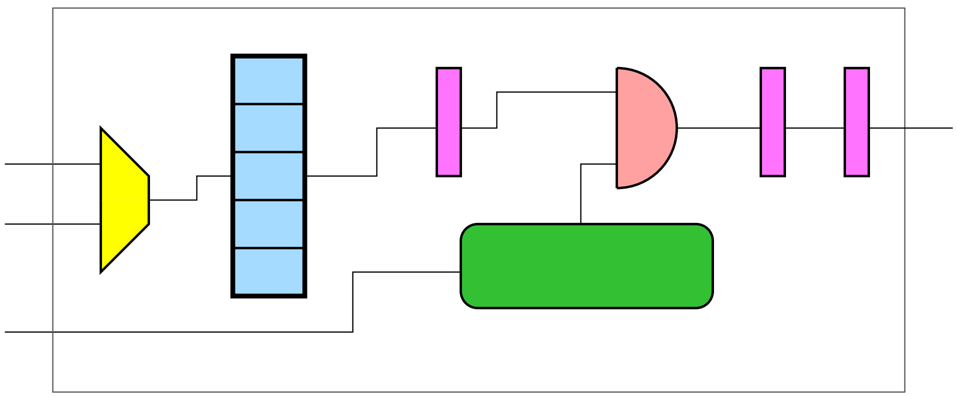
Automatic Array Abstraction [Bjesse 2008]

- Define $modeled(k)$ to mean

  "k clock cycles ago, a modeled address read was read from array"

- Use abstraction-refinement to decide values of k needed to prove property p

- The modeled addresses are chosen nondeterministically at start of each run



$modeled(3) \rightarrow p$

- Limitations
  - Many designs have unbounded latency from array read to output
  - Abstraction-refinement uses long runtimes in many examples

Semantics

1. $\sigma[\![v]\!] = \sigma(v)$, where v is a signal variable.

2. $\sigma[\![op(e_1, \ldots, e_n)]\!] =$

$$\begin{cases} OP(\sigma[\![e_1]\!], \ldots, \sigma[\![e_n]\!]), \text{ if } \sigma[\![e_i]\!] \neq \bot, \text{ for } i = 1, \ldots, n, \\ \quad \text{where } OP \text{ is the interpretation of } op \\ \bot \quad \text{if for some } i,\ \sigma[\![e_i]\!] = \bot \end{cases}$$

3. $\sigma[\![mux(e_1, e_2, e_3)]\!] = \begin{cases} \sigma[\![e_2]\!] & \text{if } \sigma[\![e_1]\!] = 0 \\ \sigma[\![e_3]\!] & \text{if } \sigma[\![e_1]\!] = 1 \\ \bot & \text{if } \sigma[\![e_1]\!] \notin \{0, 1\} \end{cases}$

4. $\sigma[\![a[e]]\!] = \begin{cases} (\sigma[\![a]\!])(\sigma[\![e]\!]) & \text{if } \sigma[\![e]\!] \in \mathrm{D}(a, \sigma) \\ \bot & \text{if } \sigma[\![e]\!] \notin \mathrm{D}(a, \sigma) \end{cases}$

5. $\sigma[\![a]\!] = \sigma(a)$, where a is an array variable.

6. $\sigma[\![write(a, e_1, e_2)]\!] =$

$$\begin{cases} (\sigma[\![a]\!])\,[\sigma[\![e_1]\!] \leftarrow \sigma[\![e_2]\!]] & \text{if } \sigma[\![e_1]\!] \in \mathrm{D}(a, \sigma) \\ \sigma[\![a]\!] & \text{if } \sigma[\![e_1]\!] \in V - \mathrm{D}(a, \sigma) \\ \mathrm{bottom}(a, \sigma) & \text{if } \sigma[\![e_1]\!] = \bot \end{cases}$$

Substates

Definition. A state σ' is called a *substate* of σ, written $\sigma' \leq \sigma$ iff

- For all signal variables s, $\sigma'(s) = \sigma(s)$, and
- For all array variables a, $\sigma'(a) \subseteq \sigma(a)$

Systems

A system $\mathcal{M}$ has the form $(\mathcal{S}, \mathcal{I}, \mathcal{N}, \mathcal{O}, \mathcal{E})$

- $\mathcal{S}$ set of state variables
- $\mathcal{I}$ set of input variables
- $\mathcal{N}$ next-state expressions $\mathcal{N} : \mathcal{S} \rightarrow expressions$
- $\mathcal{O}$ set of output variables
- $\mathcal{E}$ output expressions

Approximating Over All States

- Want to compute an overapproximate value for $\max_\sigma |\text{eindx}(e, \sigma, a)|$
- Define a function $\phi(expression, array\_variable) \rightarrow \{s_1, \ldots, s_n\}$,

  where the s_i are sets of expressions.
- We call $S = \{s_1, \ldots, s_n\}$ a two-level set.
- Each $s_i \in \phi(e, a)$ is a set of possible expressions for the values of eindx(e, σ, a)
- For all σ, $\exists s_i \in \phi(e, a)$: eindx$(e, \sigma, a) \subseteq \sigma(s_i)$
- $\forall \sigma$: $|\text{eindx}(e, \sigma, a)| \leq \|\phi(e, a)\|$,

  where $\|\{s_1, \ldots, s_n\}\| = \max_i |s_i|$, maximum size of element in $\{s_1, \ldots, s_n\}$

Definition of ϕ

Define $X \uplus Y = \{x \cup y \mid x \in X, y \in Y\}$

$$\phi(v, a) = \{\emptyset\}, \text{ if } v \text{ is a signal variable or an array variable}$$

$$\phi(c, a) = \{\emptyset\}, \text{ if } c \text{ is a constant}$$

$$\phi(b[e], a) = \begin{cases} \phi(b, a) \uplus \phi(e, a) \uplus \{\{e\}\} & \text{if } \mathrm{root}(b) = a \\ \phi(b, a) \uplus \phi(e, a) & \text{otherwise} \end{cases}$$

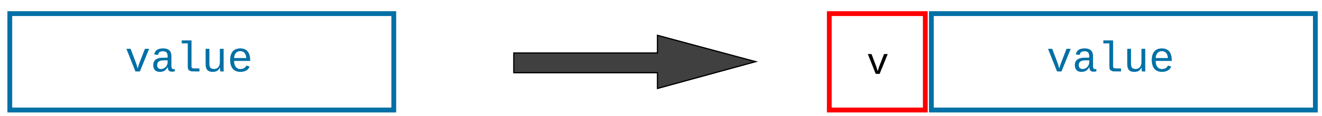
$$\phi(op(e_1, \ldots, e_n), a) = \phi(e_1, a) \uplus \ldots \uplus \phi(e_n, a)$$

$$\phi(mux(e_1, e_2, e_3), a) = (\phi(e_1, a) \uplus \phi(e_2, a)) \cup (\phi(e_1, a) \uplus \phi(e_3, a))$$

$$\phi(write(b, e_1, e_2), a) = (\phi(e_1, a) \uplus \phi(e_2, a)) \cup (\phi(e_1, a) \uplus \phi(b, a))$$

Building Abstract Model

- Original design over word-level values V $\longrightarrow$ Design over $V \cup \{\perp\}$
- Add boolean `v` field to each signal

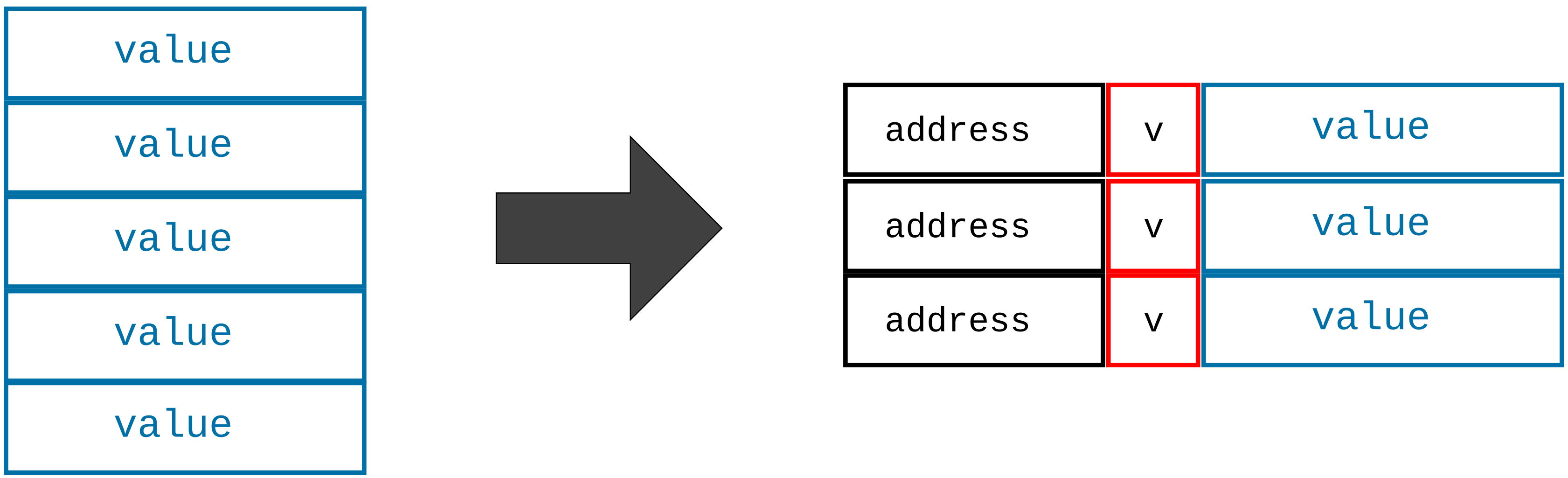


- `v = true` represents values in V; `v = false` represents $\perp$
- Concern about adding many bits to model
  - Work with word level values
- Replace blocks of combinational logic and mux with versions over $V \cup \{\perp\}$
  - Abstract models do not need to have $\perp$ version of each gate
- Safety property `p`

$$\texttt{p} \quad \Longrightarrow \quad \texttt{p.v} \rightarrow \texttt{p.value}$$

Abstract Arrays

- Each row of abstract array has `address` field and `v` field



- Address field is set nondeterministically in initial state
- Read and write operations search the address field

Early Results on Industrial Examples

- Reductions on 401 industrial examples.
- Algorithm reduced arrays in 187 examples.
- Implementation in development – some examples not fully processed.

	Reduced Number of Rows						
Original Rows	1	2	3	4	6	8	> 8
2	144						
8	1	1					
16	14	13	55				
32	37	1	25				
39	24						
48	24						
64	46	29	20	18			
128	4	158	14	23	1	11	
256	3	40	10				
1024	3		10				2

Reconfigured Arrays Example

- Reconfigured large array into two smaller arrays
- Problem is to verify sequential equivalence
- Original design has array with 1024 rows $\times$ 16 columns
- New design has two arrays, each 128 rows $\times$ 64 columns
- Array addressing, data alignment and staging logic substantially redesigned
- Design uses clock gating, so method of Bjesse does not reduce arrays