CATEGORICAL SEMANTICS OF DIGITAL CIRCUITS

Dan R. Ghica and Achim Jung
FMCAD 2016
VERITYGOS.ORG
BENEFITS OF SYNTACTIC REASONING

• formalisation
• soundness and completeness not an issue
• robustness to changes
• manipulating open terms
• partial evaluation // supercompilation
• symbolic execution // abstract interpretation
• success story in PLs: operational semantics // types // logics
CAN WE REASON EQUATIONALLY SYNTACTICALLY OPERATIONALLY ABOUT CIRCUITS?
if \( c \) then \( F(G(x)) \) else \( G(F(x)) \)
“PROPS”

$f: 4 \rightarrow 3$
COMPOSITION

\[ f : 4 \rightarrow 3 \quad g : 3 \rightarrow 1 \]

\[ f \cdot g : 4 \rightarrow 1 \]

\[ f \otimes g : 7 \rightarrow 4 \]
SOUND AND COMPLETE AXIOMS:

\textbf{STRICT SYMMETRIC TRACED MONOIDAL CATEGORY}

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DIAGRAMS WITH FEEDBACK (UP TO TOPOLOGICAL ISOMORPHISM)

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EQUATIONAL FRAMEWORK FOR CIRCUITS (NETLIST)
EXAMPLE OF EQUATION

\[(f \cdot f') \otimes (g \cdot g') = (f \otimes g) \cdot (f' \otimes g')\]
EXAMPLE OF EQUATION

\[ \text{Tr}(f \cdot (g \otimes n)) = \text{Tr}((g \otimes m) \cdot f) \]
EXAMPLE OF PROPOSITION

\[
\text{Tr}^q(g \cdot (m \otimes \text{Tr}^p(f') \otimes n) \cdot h) = \text{Tr}^{p+q}((p \otimes g) \cdot (x_{p,m} \otimes r) \cdot f' \cdot (x_{m,p} \otimes r) \otimes n) \cdot (p \otimes h)
\]

Equational $\iff$ Diagrammatic
**Streaming:** For any levels $\mathbf{v} = \mathbf{v} \otimes \mathbf{v}'$ and gate $k$, $(\delta^2 \otimes \mathbf{v}) \cdot \nabla_2 \cdot k = ((\delta^2 \cdot k) \otimes (\mathbf{v} \cdot k)) \cdot \nabla_1$. 

![Diagram](image-url)
PRODUCT : KEY PROPERTY

\[ \forall f. \]

\[
\langle f, f \rangle = \Delta_n \cdot (f \otimes f) = f \cdot \Delta_m \quad f \cdot w^m = w^m.
\]
DIAGRAMMATIC PROOF

induction
$f \cdot j = 1$
DIAGRAMMATIC PROOF

diagrammatic reasoning
DIAGRAMMATIC PROOF

induction hypothesis
DIAGRAMMATIC PROOF

diagrammatic reasoning
DIAGRAMMATIC PROOF

lemma
DIAGRAMMATIC PROOF

diagrammatic reasoning
EQUATIONS ⇒ SPECS

DIAGRAMS ⇒ CALCULATIONS
FEEDBACK + PRODUCT = “CONTROL-FLOW” ITERATION

\[ \text{iter}^n(f) = \text{Tr}^n(f \cdot (\Delta_n \otimes n)) : m \rightarrow n \]

Iteration: \( \text{iter}(f) = \langle m, \text{iter}(f) \rangle \cdot f \)
COMBINATIONAL FEEDBACK
COMBINATIONAL FEEDBACK
COMBINATIONAL FEEDBACK
COMBINATIONAL FEEDBACK

**Obs:** Unmatched delays lead to different behaviour (as they should).
GRAPH REWRITE

(submitted for publication)
GRAPH REWRITE

(submitted for publication)
GRAPH REWRITE
GRAPH REWRITE

(submitted for publication)
TRANSISTOR-LEVEL (MOSFET)
TRANSISTOR-LEVEL (MOSFET)
RELATED WORK

- **HLS**: Sheeran, Luk, Singh
- **Semantics**: Mendler, Shiple, Berry
- **Diagrammatics**: Kissinger, Coecke, Abramsky
- **Systems**: Sobocinsky, Zanasi, Bronchi
- **Category theory**: Baez, Stay, Cazanescu, Stefanescu
CONCLUSION

• the interplay of **equations** and **diagrams**
• full automation of (partial) evaluation
• a new foundation for HW modelling
• compositional VHDL/Verilog